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THE AMERICAN MATHEMATICAL MONTHLY.

EDITED BY

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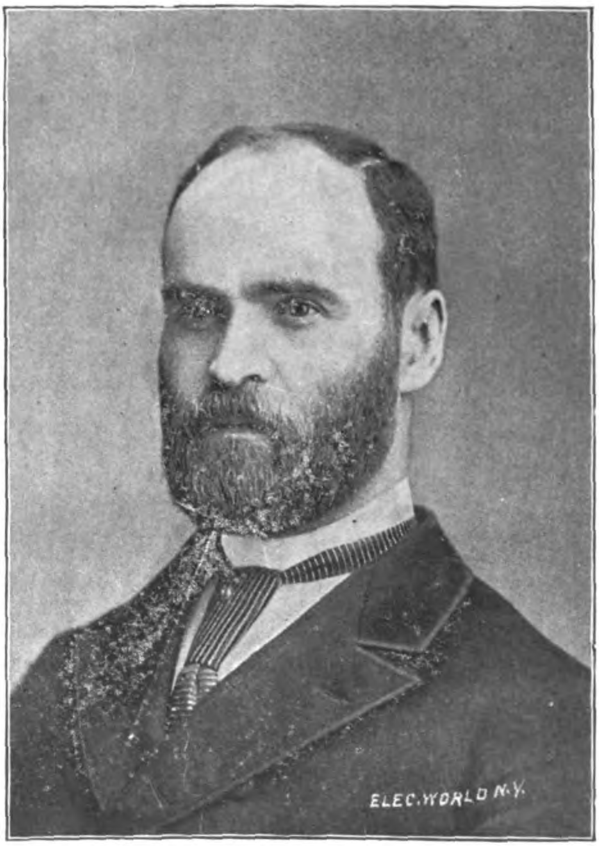


ALEXANDER MACFARLANE, M. A., D. Sc., LL. D.

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-Office at Kidder, Missouri, as Second-Class Mail Matter.

VOL. II. JANUARY, 1895. No. 1.

BIOGRAPHY.

ALEXANDER MACFARLANE, M. A., D. Sc., LL. D.

BY J. M. COLAW.

ALEXANDER MACFARLANE was born at Blairgowrie, Scotland, April 21st, 1851. He was educated at the public school, and at 13 became a regular pupil teacher in the employment of the Education Department.

In 1869, having finished his apprenticeship as a teacher and saved a little money, Mr. Macfarlane went straight to the University of Edinburgh. At that time the curriculum for Master of Arts consisted of three departments,—classical, mathematical, and philosophical; and it was customary for the more ambitious students to take the degree with honors in one of these departments. Mr. Macfarlane first entered the Junior classes in Latin and in Greek, and at the end of the session stood fourth in the former and fifth in the latter, in classes of 200, largely composed of High School graduates. He perceived that to carry himself through college it was necessary either to sacrifice a large part of his time to teaching, or else to study hard and pay his way by means of money prizes. He chose the bolder alternative. At the beginning of his second year he won in open competition the Miller scholarship, worth $400. At the end of that year he stood very high in Senior Latin and Greek and in Junior Mathematics. At the beginning of the third year he won in open competition the Spence scholarship, worth $1,000. The financial difficulty was now solved; there remained a choice of a department for honors. He was urged by the professor of Latin to go forward in the Classics, but he felt that there was more scope for originality in philosophy. In his third year he studied Senior Mathematics, Natural Philosophy and Logic. It was the custom of Professor Kelland to introduce Quaternions to his senior students. The addition of vectors was intelligible, but the product of vectors seemed to be a universal difficulty. The professor explained that in *i j* the left-hand vector was to be considered as a sort of corkscrew turning the right-hand vector through a right angle; but he did not explain how in *i i* it ceased to be a corkscrew. To get light on the subject Mr. Macfarlane bought a copy of Tait's *Treatise on Quaternions*, but found that it was addressed to mathematicians.

Before he entered the class of Logic Mr. Macfarlane was familiar with the works of Hamilton and Mill, and when a member of the class he read, at the invitation of the professor, a paper which criticised the statement of the law of Excluded Middle given by Jevons in his *Lessons on Logic*. It was his intention to study for honors in Logic and Philosophy, but perceiving how much they depended on the principles of science, and especially of exact science, he took up the advanced classes in Mathematics and Physics as a secondary study. In Experimental and Mathematical Physics he gained the highest honors and the personal friendship of Professor Tait, then, as now, the greatest figure in the University. In 1874 he was appointed Neil Arnott instructor in Physics, and in 1875 finished an unusually extensive course of undergraduate study by taking the degree of M. A. with honors in Mathematics and Physics. The University record showed that he had passed each of the seven subjects of the pass examinations with high distinction. Having, after graduation, won in a competitive examination the Maclaren fellowship, worth $1,500, he proceeded to study for the recently instituted degree of Doctor of Science. After one year spent on Chemistry, Botany, and Natural History, and two years on Mathematics and Physics, he obtained the doctorate in 1878. His thesis was an experimental research on the conditions governing the electric spark, and it was subsequently published in the *Transactions of the Royal Society of Edinburgh*. It also brought him under the notice of the celebrated electrician and philosopher, Clark Maxwell, who made various suggestions for its extension.

In 1878 Dr. Macfarlane was elected a Fellow of the Royal Society of Edinbeugh, and the first contribution which he read personally was a memoir on the Algebra of Logic. The memoir was referred by the Council to the professors of mathematics and of logic, and they reported that it was too mathematical for the one and too logical for the other to enable them to say what its value was. Dr Macfarlane enlarged the memoir and published it as a small volume under the title of *Principles of the Algebra of Logic* (1879). The volume was received with favor, and brought the author into correspondence with Munro, Jevons, Venn, Cayley, Harley, Schroeder and Halsted, who was then lecturing on the mathematical logicians at Johns Hopkins University. The main idea propounded is that of a limited and definite universe; also Euler's diagrams were further developed. In 1879 he attended the meeting of the British Association at Sheffield, and there met many of the British savants.

During 1880 Dr. Macfarlane was interim Professor of Physics at the University of St. Andrews, and in 1881 he was appointed for the usual period of three years Examiner in Mathematics in the University of Edinburgh. During these years he contributed to the Royal Society of Edinburgh a series of experimental papers on electricity, and a series of mathematical papers on the *Analysis of the Relationships of Consanguinuity and Affinity*. A paper on this subject, which he read before the Anthropological Institute of London, contains as perfect a notation for relationship as is the Arabic notation for numbers. These papers, as well as those on the Algebra of Logic, now form part of the history of Exact Logic. He also contributed to the Royal Society of Edinburgh a *Note on Plane Algebra*, which stated briefly the view he had arrived at concerning the imaginary algebra of the plane. It states that the fundamental quantity is versor rather than a vector, a view in advance of Argand's, and indeed of much that has been written more recently. By means of this algebra of the plane he deduced many series, some of which he propounded as problems in the *Educational Times* and the *Mathematical Visitor*. It was also during his tenure of office as examiner that he prepared the volume on *Physical Arithmetic*, a pioneer work, whose express object is to elucidate the logical processes involved in the application of arithmetic to physical problems.

In 1885 Dr. Macfarlane was called to the chair of physics at the University of Texas, where he became a colleague of his fellow logician, Dr. Halsted. That same year he met many of the American savants at the Ann Arbor meeting of the American Association. In 1887 he received the honorary degree of LL. D. from the University of Michigan on the occasion of their semi-centennial. His first years at the University of Texas were wholly taken up with organizing the department, but in 1889 he published as a sequel to *Physical Arithmetic* a volume of *Elementary Mathematical Tables*, distinguished for their comprehensiveness and uniformity. In 1889 he visited Paris at the time of the Exposition and met many of the continental savants at the meeting of the French Association.

On his return from Europe, he began to publish the results of his study of the algebra of space, which he approached as a logical generalization of the Algebra of the Plane. These papers are as follows: 1° *Principles of the Algebra of Physics*, read before the Washington meeting of the American Association in 1891, states the fundamental difficulties in the theory of Quaternions, lays stress on the distinction between vectors and versors, and deals mostly with the products of vectors. 2° *On the Imaginary of Algebra*, read at the Rochester meeting in 1892, gives an historical and critical account of the different interpretations of $\sqrt{(-1)}$, takes up the functions of versors, and shows that there are at least two distinct geometrical meanings of $\sqrt{(-1)}$. 3°. *The Fundamental Theorems of Analysis Generalized for Space*, contributed to the New York Mathematical Society in 1892, investigates and proves the generalized form of the Binomial and other theorems, and thus establishes the principles of spherical trigonometrical analysis. 4°. *On the Definitions of the Trigonometric Functions*, read before the Mathematical Congress at Chicago in 1893, defines these functions so as to apply to the circle, hyperbole, ellipse, logarithmic spiral, and a complex curve partly circular, partly hyperbolic. 5°. *The Principles of Elliptic and Hyperbolic Analysis*, read at the same place and time, extends spherical trigometrical analysis to the other surface of the second order. 6°. *The analytical treatment of alternating currents*, read before the International Electrical Congress at the same time, shows that plane algebra is the analysis needed for the problems of alternating currents. 7°. *On physical addition or composition*, read before the Madison meeting of the American Association in 1893, treats in a uniform manner of the composition of various physical quantities located in space, ending with the composition of screw-motions. 8°. *On the fundamental principles of exact analysis*, read before the Philosophical Society of Washington in 1894, discusses the fundamental laws of algebra, and the logical principle of generalization in analysis. 9°. *The principles of differentiation in space analysis*, recently read before the American Mathematical Society at New York, investigates the differentiation of versors, and publishes the true generalization of Taylor's theorem for space.

In 1891 Dr. Macfarlane took an active part in organizing the Texas Academy of Science, and for two years acted as its Honorary Secretary. He contributed many papers, among which may be mentioned "An Account of the Rainmaking Experiments in San Antonio," an article describing and criticising the various modern methods of rainmaking, and a paper on "Exact Analysis as the Basis of Language," where his knowledge both of languages and of mathematics comes into play.

In 1894 Professor Macfarlane resigned from the University of Texas. Throughout the nine years he labored there, he gave the new University the full benefit of his varied experience as a teacher, his accurate knowledge of University affairs, and his widespread reputation as a savant. The course in mathematical physics was so well developed as to call forth a special article in the *Rievista di Matematica*, published at Turin, Italy.

Professor Macfarlane, in addition to being a member of numerous American and British societies, is a corresponding member of the *Sociedad Cientifica Antonio Alzate*, of Mexico, and the *Circolo Matematico di Palermo*, Italy. Personally he is a characteristic Scotsman, sturdy, persevering, with a relish for hard work, thoughtful, courageous in his convictions, and endowed with more than the average share of the *perferridum ingenium Scotorum.* He is unmarried, but it is announced that in this, as in other matters, good fortune awaits him. And as he is still a young man, it is not likely that we have seen the last of his contributions to mathematical analysis.

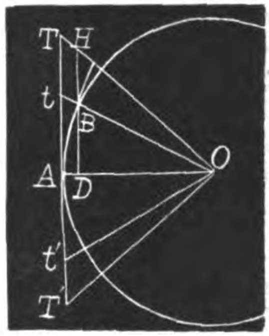
To the editors of the *Electrical World* we are indebted for the loan of the electrotype.

ISOPERIMETRY WITHOUT CURVES OR CALCULUS.

By PROFESSOR P. H. PHILBRICK, M. Sc., C. E., Lake Charles. La.

[Continued from the November Number.]

PROPOSITION IX. *If two regular polygons have the same perimeter, the one having the greater number of sides has the greatest area.*



Describe a circle with any radius $AO=r$ and circumference $2\pi r=c$.

Take AB and AC respectively the mth and nth part of a semi-circumference, and draw the secants OBt and OCT to meet the tangent AtT. Draw also DBH parallel to AT, and the tangent Be. Then, At is one-half of one side of a regular polygon of m sides, whose apothem is AO; and AT is one-half of one side of a regular polygon of n sides, whose apothem is likewise AO.

Let P=the perimeter of the polygon, the length of each side of which is $2AT$, and p=the perimeter of the polygon, the length of each side of which is $2At$.

Let a=the arc AB and A=the arc AC.

$$\text{Then, } P=AT\frac{c}{A},\quad p=At\ \frac{c}{a},\ \therefore\ \frac{P}{p}=\frac{AT}{At}\cdot\frac{a}{A}.$$

Now, $BH > Be >$ arc BC and $BD <$ arc AB.

$$\text{Dividing gives, } \frac{BH}{BD} > \frac{\text{arc } BC}{\text{arc } AB} \text{ or } \frac{DH}{DB} > \frac{\text{arc } AC}{\text{arc } AB} = \frac{A}{a}.$$

But $\frac{AT}{At} = \frac{DH}{DB}$ and therefore $\frac{AT}{At} > \frac{A}{a}$.

$$\text{Multiplying by } \frac{a}{A} \text{ we have, } \frac{AT}{At}\cdot\frac{a}{A} > 1.$$

Therefore, $\frac{P}{p} > 1$ or $P > p$.

Hence, for the same apothem, the perimeter of the polygon of the greater number of sides is the smaller.

Therefore, for *equal* perimeters, the apothem of the polygon of the greater number of sides must be the greater; and since, for equal perimeters, the areas vary as the apothem, the area of the polygon having the greater number of sides is likewise the greater.

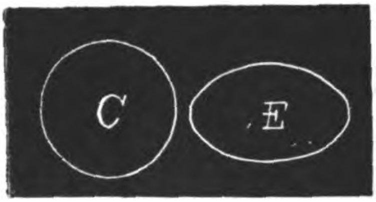
PROPOSITION X. *If two regular polygons have the same area, the one having the greater number of sides has the least perimeter.*

If the perimeters were equal, then (Prop. IX) the area of the one having the greater number of sides would be the greater. Hence, since the areas are equal, the perimeters of the polygon having the greater number of sides is the smaller.

PROPOSITION XI. *Of all isoperemetric figures, the circle has the maximum area.*

We prove by (Prop. VIII) that of regular isoperemetric polygons, that having the greatest number of sides has the greatest area, and hence if the number of sides of any regular polygon be continually increased, keeping its perimeter the same, its area will be continually increased; and as the circle is the limiting figure in conformity to which the regular polygon continually approaches, as the number of its sides is made greater and greater, the circle is that figure, which for a given perimeter contains the maximum area.

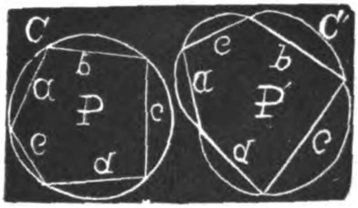
PROPOSITION XII. *Of all plane figures containing the same area, the circle has the minimum perimeter.*



Let C be a circle and E any other figure having the same area as C.

Now, by Prop. XI, if the perimeter of C was equal to that of E its area would be greater than that of E; but since it is the same, its perimeter must be less than that of E.

PROPOSITION XIII. *Of all polygons formed with the same given sides, that which can be inscribed in a circle is a maximum.*



Let the polygon P, having the sides a, b, c, d, and e, be inscribed in a circle, and the polygon P' formed with the same sides, not be inscriptible.

Upon the sides a, b, c, etc., of the polygon P' construct circular segments, equal to those standing upon the corresponding sides of the polygon P.

Then the whole figure C' thus found, has the same perimeter as the circle C.

Hence, (Prop. XI) area of $C >$ area of C'; and subtracting the circular segments from both, we have, $P > P'$.

[Concluded.]

THE INSCRIPTION OF REGULAR POLYGONS.

By LEONARD E. DICKSON, M. A., Fellow in Pure Mathematics, University of Chicago.

CHAPTER V.

[Continued from the December Number.]

II. *When the number of sides is a multiple of* 5.

In the regular 25-*gon*, $A_5 - A_{10} = 1$, being chords of the regular pentagon. But $A_1 - A_2 + A_3 - A_4 + A_5 - \ldots - A_{12} = 1$.

$\therefore \quad (A_1 - A_4 - A_6 + A_9 + A_{11}) + (-A_2 + A_3 + A_7 - A_8 - A_{12}) = 0$.

The product of these two groups expanded is seen to be 0. Hence, each group equals zero.

The sum of the chords $A_1, -A_4, -A_6, A_9, A_{11}$ equals 0; the sum of their 10 products two at a time $= 2(A_2 - A_3 - A_7 + A_8 + A_{12}) + 5(A_{10} - A_5)$ $= -5$; the sum of their 10 products three at a time $= -6(A_1 - A_4 - A_6 + A_9 + A_{11}) + 2(A_2 - A_3 - A_7 + A_8 + A_{12}) = 0$; the sum of their 5 products four at a time $= 5(A^2 - A_5) = 5$; the product of all five $= 2(A_1 - A_4 - A_6 + A_9 + A_{11}) - (A_2 - A_3 - A_7 + A_8 + A_{12}) + A_5 = A_5$. Hence, they are the roots of $x^5 - 5x^3 + 5x - A_5 = 0$.

Similarly, $A_2, -A_3, -A_7, A_8, A_{12}$ are the roots of $x^5 - 5x^3 + 5x - A_{10} = 0$.

In the regular 35-*gon*, $A_7 - A_{14} = 1$; $A_5 - A_{10} + A_{15} = 1$.

$\therefore \quad (A_2 - A_9 + A_{12} + A_{16}) + (A_6 + A_8 - A_1 - A_{13}) + (A_4 - A_3 - A_{11} - A_{17}) = 1$.

Write A, B, C for these three groups respectively.

Then $A + B + C = 1$; $AB = 3B + 3C + 4(A_{10} - A_{15})$;

$$AC = 3A + 3B + 4(-A_5 - A_{15}); \quad BC = 3A + 3C + 4(A_{10} - A_5).$$

$\therefore \quad AB + AC + BC = 6(A + B + C) + 8(-A_5 + A_{10} - A_{15}) = -2$.

$$ABC = C\{3 - 3A + 4(A_{10} - A_{15})\} = 3C - 3AC + 4C(A_{10} - A_{15}),$$

expanded, $= -1$.

Hence, A, B, C are the roots of $x^3 - x^2 - 2x + 1 = 0$. But (Chapter I.) $A_5, -A_{10}, A_{15}$ are the roots of this cubic. By inspection, or by a table of natural cosines, we determine which of the roots in the two sets correspond; viz. $A = A_5$; $B = A_{15}$; $C = -A_{10}$.

$$\therefore \quad \begin{cases} A_1 - A_6 - A_8 + A_{13} + A_{15} = 0 \\ A_2 - A_5 - A_9 + A_{12} + A_{16} = 0 \\ A_3 - A_4 - A_{10} + A_{11} + A_{17} = 0. \end{cases}$$

We may prove by our usual method that:

$A_1, -A_6, -A_8, A_{13}, A_{15}$ are the roots of $x^5 - 5x^3 + 5x - A_5 = 0$

$A_2, -A_5, -A_9, A_{12}, A_{16}$ are the roots of $x^5 - 5x^3 + 5x - A_{10} = 0$

$A_3, -A_4, -A_{10}, A_{11}, A_{17}$ are the roots of $x^5 - 5x^3 + 5x - A_{15} = 0$.

By induction, for a regular polygon of $n=5m$ sides:

$$A_1-A_{m-1}-A_{m+1}+A_{2m-1}+A_{2m+1}=0.$$

$$A_2-A_{m-2}-A_{m+2}+A_{2m-2}+A_{2m+2}=0.$$

Generally, $A_s-A_{m-s}-A_{m+s}+A_{2m-s}+A_{2m+s}=0.$

We may prove this trigonometrically by use of the formula for the sum of two cosines; but more elegantly thus:

Since $A_m-A_{2m}=1$, being chords of the regular pentagon,

$$A_s=A_s(A_m-A_{2m})=A_{m-s}+A_{m+s}-A_{2m-s}-A_{2m+s}.$$

The sum of the chords, $A_s, -A_{m-s}, -A_{m+s}, A_{2m-s}, A_{2m+s}$ equals 0; the sum of their products two at a time

$$=2(A_{2s}-A_{m-2s}-A_{m+2s}+A_{2m-2s}+A_{2m+2s})+5(A_{2m}-A_m)=-5;$$

the sum of their products three at a time$=0$; four at a time$=5$; the product of all five$=A_{5s}$. Hence, they are the five roots of $x^5-5x^3+5x-A_{5s}=0$, where s is any integer $\overline{\gtrless}\ \frac{m-1}{2}$; and $A_5, -A_{10}, A_{15}, \ldots \pm A_{5s}, \ldots$ are the $\frac{m-1}{2}$ roots of the equation (4) for the regular m-gon.

If m is not divisible by 5, one chord of every group of 5 chords is a root of this equation (4). For one and only one of the subscripts s, $m-s$, $m+s$, $2m-s$, $2m+s$ is always divisible by 5, as is seen by replacing some by their equivalents: $5m-s$, $m-s$, $4m-s$, $2m-s$, $3m-s$. The remaining four chords will be determined by a series of quadratics whose co-efficients are linear functions of the roots of (4).

Thus, if s be divisible by 5, $\pm A_s, \pm A_{2s}, \pm A_{3s}$, are roots of (4).

Then $(-A_{m-s}-A_{m+s})+(A_{2m-s}+A_{2m+s})=-A_s$; $(-A_{m-s}-A_{m+s})(A_{2m-s}+A_{2m+s})=2(-A_m+A_{2m})+(-A_{m-2s}-A_{m+2s}+A_{2m-2s}+A_{2m+2s})=-2-A_{2s}$. $\therefore$ $(-A_{m-s}-A_{m+s})$ and $(A_{2m-s}+A_{2m+s})$ are the roots of the quadratic $x^2+A_sx-(2+A_{2s})=0$.

Now $A_{m-s}\,.\,A_{m+s}=A_{2m}+A_{2s}$; $A_{2m-s}\,.\,A_{2m+s}=A_{2s}-A_m$.

The sum and product of each pair of chords being known in terms of the roots of (4), it follows that, if the $\frac{m-1}{2}$ chords of the regular m-gon be found, we can find all the chords of the regular $5\,m$-gon by solving a series of quadratics.

However, if m be divisible by 5, the five chords in any of the above groups are all, or not one of them, roots of equation (4) for the regular m-gon; for the subscripts s, $m-s$, $m+s$, $2m-s$, $2m+s$, are either all or not one of them divisible by 5, according as s is or is not divisible by 5. Hence, we can not avoid or lower the above quintic.

The regular 5-m gon depends for inscription upon the same equations *as does the regular m-gon, if m be prime to* 5; *but also upon one or more quintics of the above form, if m contains the factor* 5.

III. *When the number of sides is divisible by* 7.

For a regular polygon of $n=7m$ sides:

$$A_s - A_{m-s} - A_{m+s} + A_{2m-s} + A_{2m+s} - A_{3m-s} - A_{3m-s} = 0.$$

For, $A_m - A_{2m} + A_{3m} = 1$, being chords of the regular 7-gon.

Hence, $A_s = A_s(A_m - A_{2m} + A_{3m}) = A_{m-s} + A_{m+s} - A_{2m-s} - A_{2m+s} + A_{3m-s} + A_{3m+s}$.

By the usual method of proof, $A_s, -A_{m-s}, -A_{m+s}, A_{2m-s}, A_{2m+s}, -A_{3m-s}, -A_{3m+s}$ are the 7 roots of $x^7 - 7x^5 + 14x^3 - 7x - A_{7s} = 0$, where $s \overset{=}{>} \frac{m-2}{2}$, and $A_7, -A_{14}, A_{21}, \ldots \pm A_{7s}, \ldots$ are the roots of the equation (4) for the regular m-gon.

If m is prime to 7, one and only one chord of each of the above groups of 7 chords is a root of this equation (4).

The remaining six chords will be determined by a cubic and three quadratics, whose coefficients are linear functions of the roots of (4).

Thus, if s be divisible by 7, $\pm A_s, \pm A_{2s}, \pm A_{3s}, \ldots$ are roots of (4).

Write A for $= A_{m-s} - A_{m+s}$, B for $A_{2m-s} + A_{2m+s}$, C for $-A_{3m-s} - A_{3m+s}$.

Then $A + B + C = -A_s$; $AB + AC + BC = -4(A_m - A_{2m} + A_{3m}) + 2(-A_{m-2s} - A_{m+2s} + A_{2m-2s} + A_{2m+2s} - A_{3m-2s} - A_{3m+2s}) = -4 - 2A_{2s}$.

ABC expanded gives $3A_s + A_{3s}$. Hence, A, B, C are the roots of $x^3 + A_s x^2 - (4 + 2A_{2s})x - (3A_s + A_{3s}) = 0$.

But $A_{m-s} \cdot A_{m+s} = A_{2m} + A_{2s}$, etc. Hence, if the $\frac{m-1}{2}$ chords of the regular m-gon be found, we can find all the chords of the regular $7m$-gon by solving a cubic and 3 quadratics.

However, if m be divisible by 7, the 7 chords in any of the above groups are all or none of them roots of equation (4) for the regular m-gon. Hence, we can neither lower the above septics nor avoid them.

The regular 7m-gon depends for inscription upon the same equation as the regular m-gon, together with an additional cubic, if m be prime to 7; *but together with one or more additional septics, if m contains the factor* 7.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton), Ph. D., (Johns Hopkins), Member of the London Mathematical Society, and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the December Number.]

SCHOLION I. Here it is permitted to observe a notable difference from the hypothesis of acute angle.

For in this the general concurrence of straights cannot be demonstrated in this way, as often as any straight falling upon two, makes two internal angles toward the same parts less than two right angles; cannot, I say, be directly demonstrated, even if in this hypothesis the aforesaid general concurrence be admitted, as often as one of the two angles is right.

For although the straight *AD* be perpendicular even to the straight *AP*; in which case it certainly could not concur with another perpendicular *PL* (Eu. I. 17.); nevertheless the two angles together *DAX*, *PXA*, could be less than two right angles, in accordance with the aforesaid hypothesis, since in it the two angles together *PAX*, *PXA* may be less (P. IX.) than one right angle.

But it was worth while to have observed this. Just as, infact, solely from the admission of this general concurrence when one of the angles is right, and with an assigned incident however small, the hypothesis of acute angle can be demolished; this we will show after the three next propositions.

[To be continued.]

ARITHMETIC.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

33. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

A wine-merchant's *apparent* profit is 25% of his sales which are 10% of cost-less water. What is his actual rate per cent. of profit?

I. Solution by P. S. BERG, Apple Creek, Ohio, and the PROPOSER.

The *apparent* selling-price is $\frac{125}{100}$ of the cost. On account of the *costless* water, the *actual* selling-price is ($\frac{10}{9}$ of $\frac{125}{100}$) of 100% = $138\frac{8}{9}$%. Hence, the actual rate per cent. of profit is $38\frac{8}{9}$%.

II. **Solution by FRANK HORN, Columbia, Missouri, and Professor H. J. GAERTNER, Wilmington College, Wilmington, Ohio.**

II.
1. 100% = apparent value.
2. 125% = selling price of apparent value.
3. 90% = 100% − 10% = value of quantity sold for 125%.
4. ∴ $\frac{125}{90}$% = what 1% sells for.
5. $138\frac{8}{9}$% = 100 × $\frac{125}{90}$% = real selling price.
6. 100% = true value.
7. $38\frac{8}{9}$% = $138\frac{8}{9}$% − 100% = rate of gain.

∴ The actual rate of gain is $38\frac{8}{9}$%.

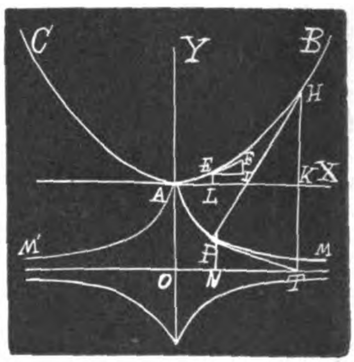
34. A chain 100m long, weighing 14 oz. to the foot, is suspended from points on a level 80m apart. What is the sag, the batter at the ends, and the horizontal tension? [From *Wentworth & Hill's High School Arithmetic.*]

Solution by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Kidder Institute, Kidder, Missouri.

The form of the chain fulfilling the conditions of the problem is the curve known as the *catenary*. Let B and C be the points of suspension of the chain, E any point in the chain, $AL=x$, $EL=y$.

Let $AE=s$ and $w=45.93$oz. the weight of a metre of length of the chain.

Then ws = the weight of the portion AE = the load suspended at E, or the vertical tension at E. Let aw = the horizontal tension at A, the weight of a units of length. Let EF be a tangent at E; then if EF represents the tension at E, EI and IF will represent the horizontal and vertical tensions respectively, at E.



Hence, $\frac{dy}{dx}=\frac{FI}{EI}=\frac{ws}{wa}=\frac{s}{a}$....(1). But $ds=\sqrt{(dy^2+dx^2)}$. ∴ $dy=\sqrt{(ds^2-dx^2)}$.

$s/a=\sqrt{(ds^2-dx^2)}/dx$, whence

$$\frac{dx}{ds}=\frac{a}{\sqrt{(a^2+s^2)}}. \quad \therefore \quad x=a\int\frac{ds}{\sqrt{(a^2+s^2)}}$$

$=a\log_e(s+\sqrt{a^2+s^2})+c$. Since $x=0$, when $s=0$, $c=-a\log a$.

$\therefore \quad x=a\log_e[(s+\sqrt{a^2+s^2})/a]$....(2). From (2), we have

$s=\frac{a}{2}(e^{\frac{x}{a}}-e^{-\frac{x}{a}})$....(3). From (1) and (3) $\frac{dy}{dx}=\frac{1}{2}(e^{\frac{x}{a}}-e^{-\frac{x}{a}})$.

$\therefore \quad y=\frac{a}{2}(e^{\frac{x}{a}}-e^{-\frac{x}{a}})+c$. Since $y=0$ when $x=0$, $c=-a$.

$\therefore \quad y=\frac{a}{2}(e^{\frac{x}{a}}-e^{-\frac{x}{a}})-a$....(4). From (3) and (4) we get $a=(s^2-y^2)/2y$.

From (2), we easily get $x=a\log_e[(y+\sqrt{y^2-a^2})/a]=a\log_e[(s+\sqrt{s^2+a^2})/a]=(s^2-y^2)/2y\log[(s+y)/(s-y)]\div\log_{10}e$.

$\therefore \log x=\log(s+y)+\log(s-y)+\log[\log(s+y)-\log(s-y)]-\log y-\log[2\log_{10}e]$.

$=\log(s+y)+\log(s-y)+\log[\log(s+y)-\log(s-y)]+\text{colog}\, y+0.0612$.

From this equation, since $x=40$m and $s=50$m, we find, by the Method of Double Position, the value of $y=26.53$m which is called the sag.

The tension at $A=wa=w\left(\dfrac{s^2-y^2}{2y}\right)=1559.78$ oz., and $\dfrac{FI}{EI}=\dfrac{a}{s}$ $=.6797$ the *batter*.

From the above equations we may obtain the four propositions as given in Wentworth and Hill's High School Arithmetic.

35. Proposed by B. F. FINKEL, Professor of Mathematics in Kidder Institute, Kidder, Missouri.

Between Sing-Sing and Tarry-Town, I met my worthy friend, John Brown,
And seven daughters, riding nags, and every one had seven bags;
In every bag were thirty cats, and every cat had forty rats,
Besides a brood of fifty kittens. All *but* the nags were wearing mittens!
Mittens, kittens—cats, rats—bags, nags—Browns,
How many were met between the towns?

[From *Mattoon's Common Arithmetic*.]

Solution by FRANK HORN, Columbia, Missouri.

II.
1. $8=$number of Browns met.
2. $8=8\times1=$number of nags.
3. $56=8\times7=$number of bags.
4. $1680=30\times56=$number of cats.
5. $67200=1680\times40=$number of rats.
6. $84000=1680\times50=$number of kittens.
7. $167888=$Browns+cats+rats+kittens.
8. $335776=167888\times2=$number of mittens worn provided that each person, cat, rat, and kitten wore one pair.
9. $636616=$Browns+nags+bags+cats+rats+mittens+kittens.

III. $\therefore$ The number of objects and persons met amounted to 636616.

NOTE.—The result given in Mattoon's Arithmetic is 218492. What interpretation did Mr. Mattoon give to the problem?—EDITOR.

PROBLEMS.

42. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

If $m=2$ct. be the interest on $M=10$ ct. for $p=10$ days, find the yearly rate per cent.

43. Proposed by B. F. BURLESON, Oneida Castle, New York.

A, in a scuffle, seized on $\frac{2}{3}$ of a parcel of sugar plums; *B* caught $\frac{3}{8}$ of it out of his hands, and *C* laid hold on $\frac{3}{10}$ more; *D* ran off with all *A* had left, except $\frac{1}{7}$ which *E* afterwards secured slyly for himself; then *A* and *C* jointly

they all went anew, for what it contained; of which, *A* got $\frac{1}{4}$, *B* $\frac{1}{3}$, and *D* $\frac{2}{7}$, and *C* and *E* equal shares of what was left of that stock. *D* then struck $\frac{3}{4}$ of what *A* and *B* last acquired, out of their hands; they, with difficulty, recovered $\frac{5}{8}$ of it in equal shares again, but the other three carried off $\frac{1}{8}$ apiece of the same. Upon this, they called a truce, and agreed that the $\frac{1}{3}$ of the whole, left by *A* at first, should be equally divided among them. How much of the prize, after this distribution, remained with each of the competitors?
set upon *B*, who, in the conflict, let fall $\frac{1}{2}$ he had, which were equally picked up by *D* and *E*, who lay perdu. *B* then kicked down *C*'s hat, and to work

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

32. Proposed by LEV. WEINER, Professor of Modern Languages, Missouri State University, Columbia, Missouri.

Find a number consisting of 6 digits which when multiplied by the first 6 natural numbers gives the same digits in rotation.

I. Solution by LEONARD E DICKSON, M. A., Fellow in Mathematics, University of Chicago.

In a memoir on "Numbers with cyclic multiples" soon to be published, I have completely discussed general problems of which this is a very special case. One of my results is that there is only one number of more than one digit which when multiplied by as many different integers as the number contains digits each product has the same digits as the original number and in the same cyclic order. This number is 142857, which answers the problem.

$$\times 1 = 142857$$
$$\times 2 = 285714$$
$$\times 3 = 428571$$
$$\times 4 = 571428$$
$$\times 5 = 714285$$
$$\times 6 = 857142.$$

Important to note is that the number $\times 7 = 999{,}999$. If in any of the above six multiples we add the number composed of the first three digits to that composed of the last three, the sum is 999.

II. Solution by the PROPOSER.

Let the digits be a, b, c, d, e, f, and let A, B, C, D, E be some one of the first 6 numbers but 1, respectively; then

$$A(10^5a+10^4b+\dots+f)=10^ra-10^{r-1}b+\dots+10^{n-r}f$$
$$B(10^5a+10^4b+\dots+f)=10^pa-10^{p-1}b+\dots+10^{n-p}f$$
$$\dots\dots\dots\dots\dots\dots\qquad\dots\dots\dots\dots\dots\dots$$
$$1(10^5a+10^4b+\dots+f)=10^5a+10^4b+\dots+f$$

$(1+A+B+\dots)(10^5a+10^4b+\dots+f)=(10^5+10^4+10^3+10^2+10^1+1)$ $(a+b+\dots+f)$.

$$\text{Now } 1+A+B+\dots=\frac{n(n+1)}{2}=21.$$

Hence, $10^5a+10^4b+\dots+e=\frac{111111}{21}(a+b+\dots+f)=5291(a+b+\dots+f)$.

By subtracting $a+b+\dots+e$ from both sides, we get $9(9999a+999b+99c+9d+e)=5290(a+b+\dots+f)$. Since the left side is divisible by 9, $a+b+c+\dots+f$ must be either 27 or 36, but 36 is readily seen to be impossible, since 5291×36 would give a number ending in 6, which when multiplied by six could not give the numbers in rotation; hence the only one to try is 27.

Now $5291\times27=142857$, and this number will be found to answer the purpose.

III. **Solution by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.**

When a common fraction in its lowest terms is changed into a decimal fraction, and this decimal fraction is a pure circulator with a full period, that is, one which begins at the first decimal and contains a number of places by one smaller than the denominator of the common fraction, then the same period will occur, only commencing at a different figure of the period, for every fraction with the same denominator, but a different numerator. The fraction $\frac{1}{7}$ produces a pure circulator with a full period, consequently, according to the principle just mentioned, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, $\frac{6}{7}$ will produce the same period, only commencing at a different figure

$\frac{1}{7}=.142857$, the next higher figure after 1 in the period is 2. $\therefore$ $\frac{2}{7}=.285714$, the next higher figure is 4. $\therefore$ $\frac{3}{7}=.428571$, etc. This answers the question proposed, the number being 142857.

IV. **Solution by H. C. WHITAKER, B. S., C. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania, and H. W. DRAUGHON, Olio, Mississippi.**

It is clear that the first digit is 1, and this can only be the last digit when the multiplier is 3 in which case the last digit of the multiplicand must be 7. Now this 7 is to be multiplied by 2, 4, 5 and 6 and hence the other digits of the required number must be 4, 8, 5 and 2. Now 8 being the largest digit must be the first digit in the product when 6 is the multiplier and hence dividing 8 by 6, we get 4 as the second digit of the required number. Now assume 8 or 5 as the third digit and multiply by 5; the digits can not be brought in the required order; hence the third digit is 2 and the number is 142857.

33. Proposed by O. E. WHITE, Trafalgar, Indiana.

Show that every algebraic equation of the nth degree, n being greater than two, which is complete in its terms may be transformed into an infinite number of equations which want their second term.

Solution by the PROPOSER.

It is shown in treatises on Higher Algebra that an equation may be changed into another equation of the same degree, but which wants its second term. Now let $x^n+mx^{n-2}+nx^{n-3}+\ldots\ldots+px^2+qx+r=0$ represent this first derived equation, and let

$(x^{n-1}+ax^{n-2}+bx^{n-3}+cx^{n-4}+\ldots\ldots+fx^2+gx+h)(x-a)=0$ represent it as factored. Equating coefficients, we have

$$m=-a^2+b\ldots.(1)$$
$$n=-ab+c\ldots.(2)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$p=-af+g\ldots.(n-3)$$
$$q=-ag+h\ldots.(n-2)$$
$$r=-ah\ldots.(n-1).$$

Now it is easily seen that we can derive without difficulty an equation in h by eliminating a from $(n-1)$ and g from $(n-2)$ and f from $(n-3)$, etc. Moreover, since $h=-\frac{r}{a}$, the derived equation in h will contain its second term.

Now, by the method by which we derived the first equation, we may derive from the equation in h another equation of the same degree wanting its second term. From this second derived equation we may derive a third, etc. Hence by continuing the process it is possible to derive an infinite number of such equations. Moreover, the first derived equation may be considered as derived from preceding equations; hence we may find an infinite number of preceding equations of the same form. By every transformation we change the value of the constant term; hence, it is possible that it may take the value zero in one of the derived equations, thus enabling us to find, at least, one root of the equation. Consider the cubic equation, and let x^3+qx+r represent its first derived equation, then, $x^3-\frac{1}{3}q^2x-\frac{2}{27}q^3-\frac{1}{4}r^2=0$ will represent its second derived equation Now, if q be negative and $\frac{2}{27}q^3=\frac{1}{4}r^2$, as in the particular case $x^3-6x+4=0$, the second derived equation reduces to the form $x^3-\frac{1}{3}q^2x=0$. Whence, $x=0,\ \pm\frac{1}{\sqrt{3}}q$, from which we easily find the roots of the first derived to be $x=-\frac{3r}{q},\ \frac{3(1+\sqrt{3})r}{2q}$, and $\frac{3(1-\sqrt{3})r}{2q}$.

It should be observed that all cubic equations that can be so resolved belong to the irreducible case.

Also solved by *H. C. Whitaker.*

PROBLEMS.

44. **Proposed by LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago.**

Find the general term in the series 1, 3, 10, 35, 126, 462, 1716, 6435, 24310, ...,which plays a remarkable part in some recent theorems in my theory of Regular Polygons.

45. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Ohio State University, Athens, Ohio.**

Find x from $\cos^{-1}\frac{1-x^2}{1+x^2}+\tan^{-1}\frac{2x}{1-x^2}=\frac{4\pi}{3}$.

GEOMETRY.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

32. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics in the Ohio State University, Athens, Ohio.**

If a conic be incribed in a triangle and its focus move along a given straight line, the locus of the other focus is a conic circumscribing the triangle.

I. **Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.**

Using trilinear co-ordinates the equation to the inscribed ellipse is of the form $\sqrt{(l\alpha)}+\sqrt{(m\beta)}+\sqrt{(n\gamma)}=0$.

Let α' β' γ', be the co-ordinates of the one focus, then

$\frac{\alpha}{\alpha'}=\frac{\beta}{\beta'}$, $\frac{\beta}{\beta'}=\frac{\gamma}{\gamma'}=\frac{\alpha}{\alpha'}$ are the equations to the lines joining it to the vertices of the triangle. The lines to the other focus make equal angles with the sides of the triangles, hence, their equations are $\alpha'\alpha=\beta'\beta$, $\beta'\beta=\gamma'\gamma$, $\gamma'\gamma=\alpha'\alpha$. $\therefore$ the co-ordinates of the other focus may be taken

$\frac{1}{\alpha'}$, $\frac{1}{\beta'}$, $\frac{1}{\gamma'}$; from this relation, if we are given the equation of any locus described by one focus, we can at once write down the equation of the locus described by the other focus.

$\therefore$ If the first focus describes the straight line $l\alpha+m\beta+n\gamma=0$, the second will describe the locus whose equation is

$$\frac{l}{\alpha}+\frac{m}{\beta}+\frac{n}{\gamma}=0, \text{ a conic circumscribing the triangle.}$$

II. **Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics in the Ohio State University, Athens, Ohio.**

In trilinear co ordinates, let the foci be $(\alpha',\beta',\gamma'),(\alpha'',\beta'',\gamma'')$. Then since the product of the perpendiculars from the foci upon tangents to a conic is constant, we should have $\alpha'\alpha''=\beta'\beta''=\gamma'\gamma''=k\ldots.(1)$.

If $l\alpha+m\beta+n\gamma=0\ldots.(2)$ be the locus of (α',β',γ'), it is plain from (1) that $\frac{l}{\alpha''}+\frac{m}{\beta''}+\frac{n}{\gamma''}=0\ldots.(3)$, or $l\beta\gamma+m\alpha\gamma+n\alpha\beta=0\ \ldots(4)$, by dropping accents, which is a circumscribing conic.

33. **Proposed by Professor B. F. SINE, Shenandoah Normal College, Reliance, Virginia.**

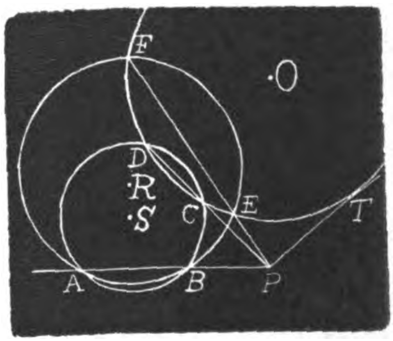
If a given circle is cut by another circle passing through two fixed points the common chord passes through a fixed point.

I. **Solution by GEORGE R. DEAN, C. E., B. Sc., High School. Kansas City, Missouri.**

The straight line containing the two given points is the radical axis of every pair of circles to which the points are common. Let the radical axis of the given circle and one of these circles intersect the given radical axis at some point O; then the radical axis of the given circle and any other circle containing the given points must pass through O, for the radical axis of three circles meet in a point.

II. **Solution by J. C. GREGG, Superintendent of Schools, Brazil, Indiania; and P. S. BERG, Apple Creek, Ohio.**

Let A and B be two fixed points and O the center of a fixed circle. Let R be the center of any circle through A and B and cutting circle O in D and C.



To show that the chord DC passes through a fixed point. Produce AB and DC to meet in P; then P is the required point. Draw the tangent PT. Then we have $PA.PB=PD.PC=\overline{PT}^2\ldots.(1)$.

Draw any other circle (center S) through A and B and cutting circle O, in two points one of which is E. Draw PE and produce it till it cuts circle S in some point X and O in F. Now from the secants PA and PX we have $PE.PX=PA.PB=\overline{PT}^2$ from (1) and from secant PF and tangent PT we have $PE.PF=\overline{PT}^2$. $\therefore$ $PE.PX=PE.PF$ and hence $PX=PF$ and the points X and F coincide and are the intersection of circles S and O

and the chord FE passes through P; and so for any circle. Q. E. D.

This problem was also solved by *G. B. M. Zerr, John B. Faught, J. F. W. Scheffer, and O. W. Anthony.*

PROBLEMS.

37. Proposed by B. F. BURLESON, Oneida Castle, New York.

Inscribe in a semicircle a rectangle having a given area: a rectangle having the maximum area.

38. Proposed by LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago.

Give a *strictly geometric* proof of my fundamental theorem on the Inscription of Regular Polygons, viz: Suppose a circle of unit radius divided at the points $A, A_1, A_2, A_3, \dots A_p, \dots$ into $2p+1$ equal parts and the diameter AO drawn. Then, if the chords $OA_1, OA_2, \dots OA_p$ be drawn, we have $OA_1-OA_2+OA_3-OA_4+OA_5-\dots\pm OA_p=1$.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

25. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

A leaf of the curve: "The Devil on Two Sticks", equation $y^4-x^4+100a^2x^2-96a^2y^2=0$, revolves around the axis of x. Deduce the expression for the volume generated.

I. Solution by the PROPOSER.

From the equation of the given curve, we deduce $y^2=48a^2\pm\sqrt{(2304a^4-100a^2x^2+x^4)}\dots(1)$; that is, $(PD)^2=48a^2+\sqrt{(2304a^4-100a^2x^2+x^4)}$, and, therefore, $(P'D)^2=48a^2-\sqrt{(2304a^4-100a^2x^2+x^4)}$. Hence the expression for the volume generated after the curve has made a complete revolution around the axis of x, becomes

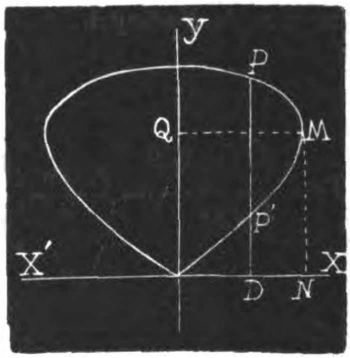
$$V=2\pi\left[\int_0^{6a}[48a^2+\sqrt{(2304a^4-100a^2x^2+x^4)}]dx-\right.$$

$$\left.\int_0^{6a}[48a^2-\sqrt{(2304a^4-100a^2x^2+x^4)}]dx\right]\dots(2).$$

Condensing (2), then factoring, etc.,

$$V=4\pi\int_0^{6a}\sqrt{(2304a^4-100a^2x^2+x^4)}\,dx=192\pi a^2\int_0^{6a}\sqrt{\left[\left(1-\frac{x^2}{36a^2}\right)\left(1-\frac{x^2}{64a^2}\right)\right]}dx\ \ldots(3).$$

I. Let $x^2/36a^2=w^2$; then will $x^2/64a^2=\frac{9}{16}w^2,=c^2w^2$, and $dx=6a\,dw$.



Making these substitutions in (3), we have

$$V=1152\pi a^3\int_0^1\sqrt{[(1-w^2)(1-c^2w^2)]}\,dw$$

$$=1152\pi a^3\int_0^1\frac{1-(1+c^2)w^2+c^2w^4}{\sqrt{[(1-w^2)(1-c^2w^2)]}}dw$$

$$=384\pi a^3\int_0^1\left[\frac{1-2(1+c^2)w^2+3c^2w^4}{\sqrt{[(1-w^2)(1-c^2w^2)]}}+\frac{2-(1+c^2)w^2}{\sqrt{[(1-w^2)(1-c^2w^2)]}}\right]dw$$

$$=384\pi a^3\left[\int_0^1\frac{1-2(1+c^2)w^2+3c^2w^4}{\sqrt{[(1-w^2)(1-c^2w^2)]}}dw+\left(\frac{1+c^2}{c^2}\right)\int_0^1\sqrt{\frac{(1-c^2w^2)}{(1-w^2)}}dw-\left(\frac{1-c^2}{c^2}\right)\int_0^1\frac{dw}{\sqrt{[(1-w^2)(1-c^2w^2)]}}\right]$$

$$=384\pi a^3\left\{\left[w\sqrt{(1-w^2)(1-c^2w^2)}\right]_0^1+\left(\frac{1+c^2}{c^2}\right)\left[\mathbf{E}(c,w)_0^1-\left(\frac{1-c^2}{1+c^2}\right)\mathbf{F}(c,w)_0^1\right]\right\}\ \ldots(4).$$

II. By making $x^2/36a^2=\sin^2\psi$, $x^2/64a^2=\frac{9}{16}\sin^2\psi,=c^2\sin^2\psi$, and $dx=6a\cos\psi d\psi$, we easily deduce from (3) the following expression:

$$V=1152\pi a^3\int_0^{\frac{1}{2}\pi}\cos^2\psi\sqrt{(1-c^2\sin^2\psi)}d\psi=384\pi a^3\left(\frac{1+c^2}{c^2}\right)\left[\mathbf{E}(c,\tfrac{1}{2}\pi)-\left(\frac{1-c^2}{1+c^2}\right)\mathbf{F}(c,\tfrac{1}{2}\pi)\right]\ldots(5),$$

which is identical with the right-hand member of (4). As a *working-result*, the right-hand member of (5) is preferable to the right hand member of (4). Expanding the Legendrian elliptic-integrals in the right-hand member of (5), uniting corresponding terms, etc., we have

$V=288\pi^2a^3[1-\frac{1}{8}c^2-\frac{1}{64}c^4-\text{etc.}]=\frac{136359}{512}\pi^2a^3\ldots(6)$. Putting $a=1$ and remembering that $\pi^2=9.8696+$, we obtain $V=2628.533+$.

II. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

The polar equation to the curve is,

$$r^2 = \frac{4a^2(24\sin^2\theta - 25\cos^2\theta)}{\sin^4\theta - \cos^4\theta} = \frac{4a^2(24\sin^2\theta - 25\cos^2\theta)}{\sin^2\theta - \cos^2\theta} = \frac{2a^2(1+49\cos 2\theta)}{\cos 2\theta}.$$

Both leaves are equal and each is symmetrical with reference to the y axis. Also, the area of the upper leaf is comprised between the limits $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$. Let A = area of this leaf, $\bar{y}$ = ordinate of its centroid. Then volume required is $V = 2\pi\bar{y}A$.

$$A = \tfrac{1}{2}\int r^2\,d\theta = a^2\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1+49\cos 2\theta}{\cos 2\theta}\,d\theta = \frac{49\pi a^2}{2}.$$

$$\bar{y} = \tfrac{2}{3}\frac{\int r^3\sin\theta\,d\theta}{\int r^2\,d\theta} = \frac{2a\sqrt{2}}{3}\cdot\frac{\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left\{\frac{1+49\cos 2\theta}{\cos 2\theta}\right\}^{\frac{3}{2}}\sin\theta\,d\theta}{\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{1+49\cos 2\theta}{\cos 2\theta}\,d\theta}.$$

$$\bar{y} = \frac{4a\sqrt{2}}{147\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left\{\frac{1+49\cos 2\theta}{\cos 2\theta}\right\}^{\frac{3}{2}}\sin\theta\,d\theta.$$

$\therefore\quad V = \frac{4\pi a^3\sqrt{2}}{3}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left\{\frac{1+49\cos 2\theta}{\cos 2\theta}\right\}^{\frac{3}{2}}\sin\theta\,d\theta.$ Let $\sqrt{2}\cos\theta = \cos\phi$;

$$\therefore\quad V = \frac{4\pi a^3}{3}\int_0^{\pi}(48-49\cos^2\phi)^{\frac{3}{2}}\operatorname{cosec}^2\phi\,d\phi = 196\pi a^3\int_0^{\pi}(48-49\cos^2\phi)^{\frac{1}{2}}\cos^2\phi\,d\phi$$

$$= 784\sqrt{3}\pi a^3\int_0^{\pi}(1-e\cos^2\phi)^{\frac{1}{2}}\cos^2\phi\,d\phi,\ \text{where } e = \frac{49}{48}.$$

$$= 784\sqrt{3}\pi a^3\int_0^{\pi}(1-\tfrac{1}{2}e\cos^2\phi - \tfrac{1}{8}e^2\cos^4\phi - \tfrac{1}{16}e^3\cos^6\phi - \tfrac{5}{128}e^4\cos^8\phi - \text{, etc.})\cos^2\phi\,d\phi$$

$$= 392\sqrt{3}\pi^2 a^3\left\{1 - \frac{3}{8}\left(\frac{49}{48}\right) - \frac{5}{64}\left(\frac{49}{48}\right)^2 - \frac{35}{1024}\left(\frac{49}{48}\right)^3 - \frac{315}{16384}\left(\frac{49}{48}\right)^4 - \text{, etc.}\right\}$$

$$= 392\sqrt{3}\pi^2 a^3\left\{1 - 3b - 5b^2 - \frac{35}{2}b^3 - \frac{315}{4}b^4 - \ldots\ldots\right\},\ \text{where } b = \frac{49}{384}.$$

The fifth term of this serie $= \frac{1}{50}$ nearly. Or thus: but since $e = c^2$ is not less than unity, let $x = \cos\phi$, then

$$\int_0^{\pi}(1-\cos^2\phi)^{\frac{1}{2}}\cos^2\phi\,d\phi = 2\int_0^1\left\{\frac{c^2x^2-1}{x^2-1}\right\}^{\frac{1}{2}}x^2\,dx = 2I.$$

Let $S = \sqrt{(c^2x^2-1)(x^2-1)}$, then $d(Sx) = \left(S + \frac{2c^2x^4 - x^2(c^2+1)}{S}\right)dx$

$$= \frac{1+3c^2x^4-2x^2(c^2+1)}{S}\,dx = \frac{1-(2c^2-1)x^2}{S}\,dx + 3dI = \frac{1-\frac{2c^2-1}{c^2}}{S}\,dx$$

$$-\frac{2e^2-1}{e^2}\,d[H(e,x)]+3dI=-\frac{e^2-1}{e^2}\,d[H'(e,x)]-\frac{2e^2-1}{e^2}\,d[H(e,x)]+3dI.$$

$$\therefore\left[x\sqrt{(e^2x^2-1)(x^2-1)}+\frac{e^2-1}{e^2}H'(e,x)+\frac{2e^2-1}{e^2}H(e,x)\right]_0^1=3I.$$

$$2I=\left[\frac{2}{3}\cdot\frac{e^2-1}{e^2}H'(e,x)+\frac{2}{3}\cdot\frac{2e^2-1}{e^2}H(e,x)\right]_0^1.$$

$$\therefore\ V=\frac{1568\sqrt{3}\pi a^3}{3e^2}\left[\frac{2e^2-1}{e^2}H(e,x)+\frac{e^2-1}{e^2}H'(e,x)\right]_0^1,\ \text{but } e^2=\frac{49}{48},$$

$\therefore\ V=\dfrac{512\sqrt{3}\pi a^3}{49}\left[50H\left(\dfrac{7}{4\sqrt{3}}\right)+H'\left(\dfrac{7}{4\sqrt{3}}\right)\right]$, where H and H' denote the hyperbolic functions corresponding to the elliptic functions E and F.

III. **Remarks by Professor J. F. W. SCHEFFER. A. M., Hagerstown, Maryland.**

This curve, called in French "la courbe du diable," is of the middle point of a chord to the equilateral hyperbola $x^2-y^2=2a^2$, the chord being of constant length and equal to seven times the transverse axis $2a\sqrt{2}$. Its equation is found thus: Let (x_1, y_1) and (x_2, y_2) be the extremities of a chord, and (x, y) any point of the curve, then we have the equations: $x^2-y^2=2a^2$ .... (1). $x^2_2-y^2_2=2a^2$ .... (2), $2x=x_1+x_2$ .... (3), $2y=y_1+y_2$ .... (4), and $(x_1-x_2)^2+(y_1-y_2)^2=392a^2$ .... (5). Subtracting (2) from (1), and considering (3) and (4), we have $(x_1-x_2)\,x=(y_1-y_2)\,y$, whence $y_1-y_2=(x_1-x_2)\frac{x}{y}$. Substituting this in (5), we get $x_1-x_2=\dfrac{14\sqrt{2}.ay}{(x^2+y^2)^{\frac{1}{2}}}$, $y_1-y_2=\dfrac{14\sqrt{2}.ax}{(x^2-y^2)^{\frac{1}{2}}}$, and combining these with (2) and (3), we get $x_1=x+\dfrac{7\sqrt{2}.ay}{(x^2+y^2)^{\frac{1}{2}}}$, $y_1=y+\dfrac{7\sqrt{2}.ax}{(x^2+y^2)^{\frac{1}{2}}}$. Substituting in (1) and simplifying we finally have $y^4-x^4-96a^2y^2+100a^2x^2=0$.

Query: Can any one furnish a reason for the peculiar name of the "devil's curve," or the name which Prof. Matz employes?

Also solved by *Prof. C. W. M. Black*.

PROBLEMS.

34. **Proposed by GEORGE LILLEY, Ph. D., LL. D., Park School, Portland, Oregon.**

A hare is at O, and a hound at E, 40 rods east of O. They start at the same instant each running with uniform velocity. The hare runs north. The hound runs directly towards the hare and overtakes it at N, 320 rods from O. How far did the hound run?

35. **Proposed by H. C. WHITAKER, B. S., C. E., Professor of Mathematics, Manual Training School, Philadelphia, Pa.**

Water is running into a vessel in the shape of a frustum of a cone (radii upper and lower bases 15 inches and 10 inches, respectively, and altitude 20 inches) at the rate of 10 cubic inches per second. When the depth is 8 inches at what rate is it increasing?

MECHANICS.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

14. Proposed by ALFRED HUME, C. E., Sc. D., Professor of Mathematics, University of Mississippi, University P. O., Miss.

"The center of a sphere of radius C moves in a circle of radius A and generates thereby a solid ring, as an anchor-ring: prove that the moment of inertia of this ring about an axis passing through the center of the direct circle and perpendicular to its plane is $\frac{1}{4}\pi^2\delta ac^2(4a^2+3c^2)$."

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

If the moment axis be the axis of z, the origin being the center of the ring, and the axis of x and y any two diameters at right angles the required moment could be obtained from $\int\int(x^2+y^2)zdxdy$, having the equation to the surface of the ring.

But the following method quoted by Williamson in the Int. Cal., New York Edition, 1884, art. 212 from Townsend is so concise I prefer to give it.

Let y, Y be the distances of any point in the meridian section of the sphere from that diameter of the section parallel to the moment axis, and to the moment axis. Then if dA be the element of area of the generating section, the mass of the elementary ring generated by dA is $2\pi\mu YdA$, and the moment of inertia of this ring is $2\pi\mu Y^3dA$.

$\therefore$ the required $M.I.=2\pi\mu\int Y^3dA=2\pi\mu\int(a+y)^3dA$

$$=2\pi\mu\int(a^3+3a^2y+3ay^2+y^3)dA\;\ldots\ldots(1).$$

But from theory, $\int ydA=0$, $\int y^3dA=0$, and if k be the radius of gyration of the generating section, $\int y^2dA=Ak^2$; then (1) becomes

$M.I.=2\pi\mu aA(a^2+3k^2)=2\pi^2\mu ac^2(a^2+\frac{3}{4}c^2)$

$=\frac{\pi^2\mu ac^2}{2}(4a^2+3c^2)\ldots.(2)$, in which μ is used instead of δ, and the result is twice as great as given in the statement of the problem.

The advantage of this method lies in the fact that it is general for A and k^2, which are therefore the only quantities to be worked out before setting down the special result.

II. Solution by F. P. MATZ, M Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

According to well-known principles $V=\pi a^2\times 2\pi c=2\pi^2ac^2$, $M=V\delta=2\pi^2\delta ac^2$, and the Radius of Gyration$=X=\sqrt{(\frac{1}{2}a^2+\frac{3}{8}c^2)}$. Hence the required Moment of Inertia, *Nystrom's Mechanics*, becomes $E=MX^2$ $=\frac{1}{4}\pi^2\delta ac^2(4a^2+3c^2)$.

III. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics in the University of Mississippi.

If the center of the generating circle be taken as the origin and a perpendicular from this point to the axis of revolution as the axis of x, the equation of the moving circle is $x^2+y^2=c^2$.

Divide the ring formed into layers of infinitesimal thickness, dy, by planes parallel to the plane of the director circle.

The moment of inertia of any layer whose external radius is $a+x$ and internal $a-x$ is $\left[\frac{\pi}{2}\rho(a+x)^4-\frac{\pi}{2}\rho(a-x)^4\right]dy$, ρ being the density.

Therefore the moment of inertia of the entire ring is

$4\pi\rho a\int_{-c}^{c}(a^2+c^2-y^2)(c^2-y^2)^{\frac{1}{2}}dy$, substituting c^2-y^2 for x^2.

Performing the integration the result is $\frac{\pi^2\rho ac^2}{2}(4a^2+3c^2)$ which is double that given by Price.

This problem was also solved by *W. Wiggins, G. B. M. Zerr, and P. H. Philbrick.* Their solutions will be published next month.

PROBLEMS.

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20. Proposed by CHAS. E. MYERS, Canton, Ohio.

A flexible cord of given length is suspended from two points whose coordinates are (x, y) and (x', y'). What must be the condition of the cord in order that it may hang in the arc of a circle?

21. Proposed by J. A. CALDERHEAD, Superintendent of Schools, Limaville, Ohio.

Show that, in the wheel and axle, when a force P, acting at the circumference of the wheel, supports a weight Q upon the axle,

$$P.(R\mp\rho\sin\varepsilon)=Q(r\pm\rho\sin\varepsilon)\pm W\rho\sin\varepsilon,$$

where R, r, and ρ are the radii of the wheel, the axle, and their common axis respectively, and ε is the limiting angle of resistance.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

16. Proposed by H. W. DRAUGHON, Olio, Mississippi.

Find three numbers such that the cube of any one plus the sum of the squares of the other two, will be a square.

Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Let ax, bxy, and cxz represent the required numbers; then we have $a^3x^3+(b^2y^2+c^2z^2)x^2=\square \dots (A)$, $b^3x^3y^3+(a^2+c^2z^2)x^2=\square \dots (B)$, and $c^3x^3z^3+(a^2+b^2y^2)x^2=\square \dots (C)$. Omitting in (A), (B), and (C), the factor x^2, and putting $x=2bcyz/a^3$, we have (A) as a perfect square.

Substitute this value of x in the first term of (B); then, obviously, the condition that (B) will be a perfect square, is $2b^4cy^4z/a^3=2acz$.

$\therefore y=a/b \dots (1)$. After performing a similar operation in (C), we obtain $2bc^4yz^4/a^3=2aby$. $\therefore$ $z=a/c \dots (2)$.

Consequently $x=2bcyz/a^3=2/a \dots (3)$; and the required numbers are $ax=2$, $bxy=2$, and $cxz=2$.

[NOTE.—Can any of our contributors find three *unequal* numbers answering the conditions of this problem? The proposer and several contributors have reported that they had as yet failed to solve it. The problem seems difficult of solution, or at least the EDITOR does not now see any way clear to a solution of it.]

17. Proposed by ARTEMAS MARTIN, LL.D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Is it possible to find two positive whole numbers such that each of them and also their sum and difference, when diminished by unity shall all be squares?

Solution by the PROPOSER.

Let x^2+1 and y^2+1 denote the numbers required; then their sum $=x^2+y^2+2$, their difference $=x^2-y^2$, and we have

$$x^2+y^2+1=\square=v^2 \ldots (1),$$
$$x^2-y^2-1=\square=w^2 \ldots (2).$$

From the first of these equations, $y^2=v^2-x^2-1 \ldots (3)$. Adding (1) and (2) we get $2x^2=v^2+w^2 \ldots (4)$. Let $v=t+u$, $w=t-u$ and (4) becomes $x^2=t^2+u^2 \ldots (5)$, which is satisfied by

$t=p^2-q^2$, $u=2pq$, $x=p^2+q^2$, and then $v=p^2+2pq-q^2$.

Substituting these values of x and v in (3) we get

$$y^2=4pq(p^2-q^2)-1 \ldots (6).$$

As the right hand member of (6) is of the form $4m-1$ it can not be an integral square, and therefore the problem is impossible.

II. **Solution by C. A. ROBERTS, Long Bottom, Ohio.**

Let (x^2+1) and (y^2+1) be the numbers, which when diminished by 1 give x^2 and y^2.

Let $(x^2+1)-(y^2+1)-1=a^2=x^2-y^2-1$, which call (I);

Let $(x^2+1)+(y^2+1)-1=b^2=x^2+y^2+1$, which call (II).

The square root of an even square, is even; the square root of an odd square, is odd. Take (I), $a^2=x^2-y^2-1$, and transposing, $a^2+y^2+1=x^2$ or three squares whose sum is a square. The square of any even number is divisible by 4, without a remainder, and is therefore said to be of the form of $(4n)$. If the square of any odd number be divided by 4, there will be a remainder of 1, and such squares are said to be of the form of $(4n+1)$. Any number not of the form of $(4n)$ or $(4n+1)$ is not a square. Let us determine in the equation $a^2+y^2+1=x^2$ whether a and y, are both odd, both even, or one of them odd and the other even.

1 is odd and of the form of $(4n+1)$; if a is even and y odd, or if y is even and a odd, we have for the form of the sum $(4n)+(4n+1)+(4n+1)$ $=(12n+2)$, which is of the form of $(4n_1+2)$ and which can not be a square. If a and y are both odd, we have for the form of the sum, $(4n+1)+(4n+1)$ $+(4n+1)=12n+3$, which is of the form of $(4n_1+3)$, and which can not be a square. If a and y are both even, we have for the form of the sum, $(4n)+(4n)+(4n+1)=(12n+1)$, which is of the form of $(4n_1+1)$, and may be a square, and if $(4n_1+1)$ is a square, as x^2, it is an odd square, and x is odd. Therefore in order that the equation $a^2+y^2+1=x^2$ shall be true in integers, a and y *must* be even numbers, and x *must* be an odd number.

Take (II) $x^2+y^2+1=b^2$; applying to (II) the reasoning in (I) x and y *must* be even, and b odd, or x must be both odd, (as in I) and even, (as in II). As this is impossible with the same value of x, there are no such numbers as called for in the problem.

[NOTES.—CHAS. DE MEDICI of 60 West 22nd St., New York, overlooking the punctuation of the problem as published, gives 64 and 81 as the numbers, and adds an interesting exhibit showing that the curio of these values are not by any means limited to what the question, as he read it, asked for.

M. A. Gruber, P. H. Philbrick, and G. B. M. Zerr, should have been credited for solving problem 15, December Number. Their solutions were selected for publication, but owing to the fact the December Number had to be cut short in order to get it out without further delay their solutions were omitted.—EDITOR.]

PROBLEMS.

25. **Proposed by M. A. GRUBER. A. M., War Department, Washington, D. C.**

Find, if possible, integral values of each of the seven linear measurements of a rectangular parallelopiped; i. e. length, breadth, height, the diagonals of each of the three different rectangular sides, and the diagonal from an upper corner to the opposite lower corner; or, find integral values, if possible, of a, b, c, d, e, f, and g, as shown in the equations,—$a^2+b^2=c^2$, $a^2+d^2=e^2$, $a^2+f^2=g^2$, $b^2+d^2=f^2$, $b^2+e^2=g^2$, $c^2+d^2=g^2$, $c^2+e^2=f^2$. If not possible, how many of them can have integral values? and which?

26. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

Find (1) a *square fraction* the arithmetical difference of whose terms is a *cube*; and (2) find a *cubic fraction* the arithmetical sum of whose terms is a *square*.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

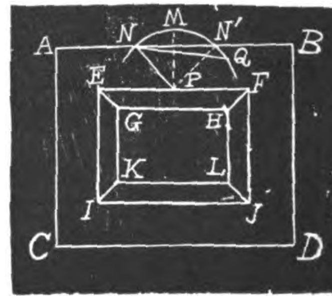
12. **Proposed by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.**

A large plane area is ruled by two sets of parallel equidistant straight lines, the one set perpendicular to the other. The distance between any two lines of the first set is a; the distance between any two lines of the second set is b. If a regular polygon of $2n$ sides be thrown at random upon this area, find the chance that it will fall across a line, the diameter of the circum-circle of the polygon being less than a or b.

II. **Solution by H. W. DRAUGHON, Clinton, Louisiana.**

In the rectangle $ABCD$ let $AB=a$ and $AC=b$. Let c=apothem of polygon, and r=radius of its circum-circle.

Let the sides of the rectangle $EFIJ$, be parallel to, and distant c, from the corresponding sides of $ABCD$, and let the sides of the similarly placed rectangle $GHKL$, be r distant from the corresponding sides of AB. Draw EG, FH, LJ, and IK.



1. If the center of the polygon falls without the rectangle $EFIJ$, the polygon will cross a line in every possible position. $\therefore$ the number of favorable positions for this case is, $n_1 = 2\pi r \times$ area of surface on which center falls $= 2\pi r[ab-(a-2c)(b-2c)]$. 2. Let us suppose that the center falls on the point P within the trapezoid $EFGH$. From P as a center, radius r, draw an arc cutting AB in the points N and N'. Draw one side of polygon NQ, also draw PM, perpendicular to AB. Put $AM=x$ and $PM=y$. The number of favorable positions for the point P, is obviously, $2n \times \text{arc} NN' = 4nr\cos^{-1}\left(\frac{y}{r}\right)$.

When x varies from $a-r$ to r, y can have any value from r to c. When x varies from r to c, y can have any value from x to c. The integration between the remaining limits for x and y, will obviously give the same result as that between last mentioned limits.

$\therefore$ in this case, the total number of favorable positions is,

$$n_2 = 4nr\left[2\int_c^r\int_c^x \cos^{-1}\left(\frac{y}{r}\right)dxdy + \int_r^{a-r}\int_c^r \cos^{-1}\left(\frac{y}{r}\right)dxdy\right]$$

$$=4nr\left[2\int_c^r \left(x\cos^{-1}\left(\frac{x}{r}\right)dx - \sqrt{(r^2-x^2)}dx - c\cos^{-1}\left(\frac{c}{r}\right)dx\right.\right.$$

$$\left.\left. + \sqrt{(r^2-c^2)}\,dx\right) + \int_r^{a-r}\left(-c\cos^{-1}\left(\frac{c}{r}\right)dx + \sqrt{(r^2-c^2)}\,dx\right)\right]$$

$$=4nr\left\{2\left[\left(\frac{x^2}{2}-\frac{r^2}{4}\right)\cos^{-1}\left(\frac{x}{r}\right) - r^2\cos^{-1}\left(\frac{x}{r}\right) + \frac{x}{2}\sqrt{(r^2-x^2)}\right.\right.$$

$$\left.\left. + \frac{r^2}{2}\cos^{-1}\left(\frac{x}{r}\right)\right]_c^r + \left[-cx\cos^{-1}\left(\frac{c}{r}\right) + x\sqrt{(r^2-c^2)}\right]_r^{a-r}\right\}$$

$$=4nr\left[\left(\tfrac{3}{2}r^2+2cr-ac-c^2\right)\cos^{-1}\left(\frac{c}{r}\right)+(a-2r-c)\sqrt{(r^2-c^2)}\right].$$

If the center falls within rectangle $GHKL$, the polygon can not cross a line.

3. The total number of favorable positions, when the center falls within the trapezoid $GKEI$ is found by changing a to b in the value of n_2. $\therefore$ we have for this case, number of favorable positions,

$$n_3 = 4n\left[\left(\tfrac{3}{2}r^2+2cr-bc-c^2\right)\cos^{-1}\left(\frac{c}{r}\right)+(b-2r-c)\sqrt{(r^2-c^2)}\right].$$

4. If the center falls within the trapezoids $KLIJ$ or $HLFJ$, the number of favorable positions is, respectively, n_2 and n_3. $\therefore$ when the center falls within rectangle $ABCD$, the number of favorable positions is,

$n_4 = n_1 + 2n_2 + 2n_3 = 2\pi r[ab-(a-2r)(b-2r)]$

$$+8nr[(\tfrac{3}{2}r^2+2cr-ac-c^2)\cos^{-1}\left(\frac{c}{r}\right)+(a-2r-c)\sqrt{(r^2-c^2)}]$$

$$+8nr[(\tfrac{3}{2}r^2+2cr-bc-c^2)\cos^{-1}\left(\frac{c}{r}\right)+(b-2r-c)\sqrt{(r^2-c^2)}]$$

$$=2\pi r[ab-(a-2r)(b-2r)]+8nr[(3r^2+4cr-(a+b)c-2c^2)$$

$$\cos^{-1}\left(\frac{c}{r}\right)+(a+b-4r-2c)\sqrt{(r^2-c^2)}].$$

Let S be the number of rectangles, ab, then the probability required is, $P=Sn_4\div 2\pi rSab=n_4\div 2\pi rab=\{\pi[ab-(a-2r)(b-2r)+4n[(3r^2+4cr$

$$-(a+b)c-2c^2)\cos^{-1}\left(\frac{c}{r}\right)+(a+b-4r-2c)\sqrt{(r^2-c^2)}]\}\div\pi ab.$$

PROBLEMS.

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24. **Proposed by F. P. MATZ, M. Sc. Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

The average area of the triangle formed by three perpendiculars drawn from the sides of the triangle (a, b, c), is $\mathbf{A}=(a^4+b^4+c^4)/48\triangle$.

25. **Proposed by G. B. M. ZERR, Principal of High School, Staunton, Virginia.**

The probability that the distance of two points taken at random in a given convex area A shall exceed a given limit (a) is

$$\triangle=\frac{1}{3A^2}\iint(C^3-3a^2C+2a^3)\,dp\,d\theta,$$

where C is a chord of the area, whose co-ordinates are p, θ; the integration extending to all values of p, θ, which give a chord $C>a$. What is $\triangle$ when the area is a circle? If in the circle $a=r=$radius $\triangle=\dfrac{3\sqrt{3}}{4\pi}$.

INFORMATION.

PROFESSOR ARTHUR CAYLEY DEAD.

The Distinguished English Mathematician Passes Away at Cambridge.

LONDON, Jan. 31.—Prof. Arthur Cayley, of the University of Cambridge, died to-day, in the seventy-fourth year of his age. He had been for thirty-two years Sadlerian professor of pure mathematics at Cambridge, and was regarded by educators everywhere as one of the three greatest of co-temporary mathematicians, the others being Professor Sylvester, of Oxford, and Professor Klein, of Goettingen.

Before becoming a professor at Cambridge he had been for fourteen years a conveyancer at Lincoln's Inn, London. He had been educated at King's College, London, and at Trinity College, Cambridge, where he graduated as senior wrangler in 1842, and was shortly after elected a fellow.

The scientific writings of Professor Cayley relate to every branch of pure mathematics beside dynamics and astronomy.

The honors which the dead mathematician received from his own university and other universities, from English and foreign societies and from foreign governments can be numbered by the score and include honorary degrees, medals, fellowships and other honorary positions. The French government in 1890 made him an officer of the Legion of Honor, and he was also a correspondent of the French Institute and the winner of the Copley and Royal medals from the Royal Society, of which he is a fellow.

The higher education of women was one of the many practical problems in which he was interested, and for a number of years he had been chairman of the association to which belongs Newnham College, at Cambridge.

AT THE JOHNS HOPKINS UNIVERSITY.

By request of Professor J. J. Sylvester, Professor Arthur Cayley, the Sadlerian professor of pure mathematics of Cambridge, England, was associated in the mathematical work of the Johns Hopkins University, from January to June, 1882. Professor Sylvester, now the Savilian professor of Geometry in the University of Oxford, England, was then at the head of the Department of Mathematics in the Johns Hopkins University; and the presence in Baltimore, at the same time, of two of the most distinguished of the world's mathematicians, attracted much attention throughout the country. Professor Cayley was present at a number of university receptions given in his honor, and was also widely entertained in a social way. His portrait at the university was placed in a prominent position yesterday when the news of his death was received, and appropriately draped with black.—F. P. MATZ.

P. L. Tchebichef, member of the St. Petersburg Academy of Science, died Dec. 8th last, at the age of 74. Dr. Halsted has written a biography of this great mathematician, which will appear in the March number of the MONTHLY.

On February 1, G. H. Harvill, Tyler, Texas, will begin the publication of a Monthly School Journal, "*The Investigator.*" This new journal is to be devoted to Elementary Mathematics, English Grammar, History, Geography, Philosophy, etc. The price of the Journal will be $1.00 per year. The editors of the MONTHLY extend a hand of welcome to this new periodical and wish it abundant success.

Dr. George Bruce Halsted has just received a letter from Dairoku Kikuchi, Member of the House of Peers of Japan, Professor of mathematics in the Imperial University of Tokyo, Japan, in which he says:

"I desire to acknowledge with many thanks your translation of Professor Vasiliev's Address. The Russian original I received some time ago, but I regret that I could not read it. Your Bolyai ought to have been printed here long ago, but owing to unavoidable circumstances has been delayed. It will be ready, however, in a week from now, and then I shall send you some copies at once.

We are fighting the battle of Civilization against the Chinese, and with easy success which even the most sanguine of us had scarcely expected before the war began."

[It may be remembered that the Japanese issued for their own use, in English, a beautiful edition of Dr. Halsted's Lobachevsky, and asked permission to issue his Bolyai.]

The publication of the weekly journal *Science* was resumed January 1st, 1895, under the charge of an editorial committee, of which Simon Newcomb represents mathematics. Professor J. McKeen Cattell of Columbia College, Editor of the Psychological Review, in asking Dr. Halsted to give an account of an Italian and German work, adds: "You are probably the only eminent man of Science in America who reads Russian, so any report of recent scientific work from Russia would be very acceptable." The MONTHLY is proud to have the illustrious name of Dr. Halsted on its list of contributors.

EDITORIALS.

THIS number of the MONTHLY was mailed Feb. 13. The February number will be issued as soon as possible.

DR. HALSTED'S article on Non-Euclidean Geometry was cut short this month owing to sickness in his family. February MONTHLY will contain the usual number of pages devoted to that wonderful department of mathematics.

THE last two lines of problems 43. pp. 12 and 13, should go to the top of page.

WHAT THEY SAY OF THE MONTHLY.—W. I. Taylor, Instructor in Mathematics, Baldwin University, Berea, Ohio, says: I am so well pleased with the MONTHLY and have received so much help from it, that I feel that I ought not to do without it....Dr. Alexander Macfarlane, Ithaca, N. Y., says: The December number of the MONTHLY has just arrived to-day. I congratulate you upon the success of the first volume. I shall endeavor to give you what assistance I can and I think that the MONTHLY deserves the support of the professors and teachers of mathematics.

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BIOGRAPHY.

PROFESSOR WILLIAM CHAUVENET.

BY F. P. MATZ, M. SC., PH. D., NEW WINDSOR, MARYLAND.

"Professor William Chauvenet ranks among the *coryphæi* of science in America. He and Professor Benjamin Peirce have done more for the advancement of mathematical and astronomical science, and for the *raising to a higher level* of the instruction in these subjects, than any other two Americans. It is our wish, on that account, to place before the reader a somewhat full sketch of the life and works of Professor William Chauvenet."

"William Marc Chauvenet, the father of the subject of this sketch, was born at Narbonne, France, in 1790, and came to the United States in 1816. He was the youngest of four brothers, another of whom also came to this country but has left no descendants. William Marc was a man of education and culture, versed in several languages, and a constant reader. He came to America, however, in connection with a manufacturing enterprise which had its headquarters in New York, with a branch at Boston. The latter department was under Mr. Chauvenet's charge, and here he married, in 1819, Miss Mary B. Kerr, of Roxbury, Mass. This was prior to the occurrence of a heavy defalcation in the New York house, which broke up the enterprise so badly that all investments in it proved to be total losses. Mr. Chauvenet having an idea that rural life would suit his taste, bought a small farm close to *Milford, Pike County, Pennsylvania*, and it was here that his only child, WILLIAM CHAUVENET, was born, May 24, 1820.

By the advice of friends Mr. Chauvenet soon gave up his attempt at farming, and settled in Philadelphia, where his son grew to manhood. His rapid progress at school attracted such attention from his instructors, especially in mathematics, that his father easily yielded to their advice, and sent him to Yale College, where he graduated in 1840, '*facile princeps*' in mathematics, and *high in standing* in all other branches. The honorary societies, 'Phi Delta Kappa' and 'Chi Delta Theta,' denoting respectively the fifteen of highest standing and the fifteen best writers of the class, each claimed him as a member.

Upon his return to his home he was, after a brief incumbency in a subordinate position, appointed professor of mathematics in the Navy. Late in 1841 he married Miss Catherine Hemple, of Philadelphia. Shortly after this he served a brief term on a United States vessel, as instructor to midshipmen, but did not go upon a foreign cruise, and was soon detailed to the 'Naval Asylum,' then situated at Philadelphia. Here midshipmen were sent at that time, to receive instruction and examinations, principally in mathematics and the theory of navigation. The young professor was struck with the imperfections in the education of naval officers, and it was very largely through his efforts, aided by such influences as he could bring to bear on the matter, that a commission was appointed *to draft a plan* for a fixed 'Naval Academy,' corresponding to the Military Academy at West Point. Six naval officers constituted this commission, Professor Chauvenet being one of the number. The appointment of so young a man (he was but twenty four at the time) on a commission of such importance indicates what must have been his record, and the impression he made upon his seniors in years and rank.

The Naval Academy was formally called into existence in the year 1845, being located at Annapolis, Md. Professor Chauvenet was appointed to the chair of mathematics, and resided at the academy until his resignation from the Navy in 1859.

It was not long after this change of residence that he began to plan his work on trigonometry, which was published in 1850. Its title, 'A Treatise on Plane and Spherical Trigonometry,' partly indicated that it was not a students' class-book merely, but that it took up most of the more advanced applications of the subject. It soon assumed the position it still retains as the standard reference work in its line.

Some time before this publication, Professor Chauvenet had persuaded his father to retire from business and accept a position at the academy. He came as instructor in the French language, and remained at his post until his death in 1855.

It having been decided to erect an astronomical observatory at the academy, Professor Chauvenet was made professor of astronomy and put in charge of the observatory. As he became more and more interested in his work, the idea of his next treatise, 'Spherical and Practical Astronomy,' grew upon him, and, just previous to his resignation, had assumed such form that he issued a prospectus for its publication as a subscription work. This was never carried out.

In 1859 he was notified that his application for the professorship of mathematics at Yale College would be followed by his election to that position.

Almost simultaneously with this came a call to St. Louis, Mo., where he was offered the same chair in the then newly established Washington University. After much deliberation he accepted the latter, and removed with his family (including at that time his mother) to St. Louis, in the fall of 1859.

Chancellor Hoyt, who was at the head of the 'Washington' at this time, died early in the 'sixties,' and Professor Chauvenet was elected to the vacancy. He still continued his duties as professor of mathematics, and also *resumed* his work on the 'Astronomy.' The risks of publication were great, and his means did not enable him to guarantee the publishers against loss. The Civil War was in progress, and the time seemed inopportune for such an undertaking. It was to the liberality of certain friends, chiefly to the initiative of Mr. (afterward Judge) Thomas T. Gantt, of the St. Louis bar, that a guarantee fund was raised, sufficient in the opinion of the publishers to prevent any loss to them. The work, in two octavo volumes, was published in 1863.

Few works of a scientific nature, by American authors, have been received with such universal favor, by those competent to judge of its merits, as was this. Its reputation was quite as great in Europe as here, while of course it is not (as it was never intended to be) a treatise much known outside of scientific, and more especially *astronomical*, circles. Its scope, and the rigorous methods adopted, are sufficiently indicated in the author's preface. It retaines to day its standard character, as fully as when this was first recognized by the scientific world upon its publication.

Professor Chauvenet's mother died in St. Louis, not long after the appearance of the Astronomy, and it was but a few months later that the first symptoms of the disease that proved finally fatal to him, made their appearance. Partial recovery and resumption of his duties was followed by a long period of alternating hopes and fears, during which time he tried in vain different parts of the United States, from South Carolina to Minnesota. During this illness he worked at his *only* elementary publication, the 'Geometry,' which he undertook, partly because he had long thought that the popular texts of the day were marked by too strict an adherence to strictly 'Euclidian' methods, and partly because he wished to provide an income for his family, by the publication of a text for which he had reason to suppose there would be a larger sale than was possible with advanced treatises. The publication of this work shortly *antedated* his death, which occurred at St. Paul, Minn., December 13, 1870.

Professor Chauvenet left, so to speak, two distinct impressions behind him. By far the larger circle, in numbers, of those who knew him, were of those to whom his scientific attainments, though known, were as traditions merely, since they were in a field whose extent was to them only a matter of vague conjecture. To these he left the impression of a man of wide and varied culture, and keen critical taste. Probably few scientists of distinction were more keenly interested in lines outside of their own specialties. He was not only a *critic* in music, but to his latest day a *pianist* of no mean ability, always expressing a preference, in his own playing, for the works of Beethoven, which he rendered with an interpretation which never failed to excite the admiration of musicians whose execution surpassed his own. His knowledge of English literature was extensive, but he read and re-read a few authors, at least in the latter part of his life, and his great familiarity with many of these gave point to the old adage, 'fear the man of few books,' though perhaps not in the sense in which these words were originally intended. He was a ready writer, and contributed at times, reviews, partly scientific, to various journals. His style was clear and unaffected, while, in the review of a pretentious or ignorant author, he had the gift of a delicate sarcasm, so light at times as only to be visible to one reading between the lines For other pretenders he could drop this mask, and write with severity; but only twice in his life, to the knowledge of the present writer, did he ever do so. In addition to his more important writings, he was the author of a 'Lunar Method,' still used in the Navy, and invented a device called the 'great circle protractor,' by which the navigator is enabled (knowing his position) to lay down his course on a 'great circle' of the globe, without further calculation. This invention was purchased by the United States Government not long after the close of the Civil War.

Professor Chauvenet's scientific reputation needs little comment on the part of the present writer. He was one of a group of scientists in his own or cognate lines, who were the first to secure recognition abroad, as well as at home, for the position of the exact sciences in the United States. Among his more intimate scientific friends were Benjamin Peirce and Wolcott Gibbs (Harvard), Dr. B. A. Gould, and many others whose names are as household words in the history of scientific progress in this country. At the *formation* of the National Academy of Sciences he was one of the prominent members. But while his scientific reputation will outlast his personal memory, it is doubtful if to those who knew him, even of his scientific associates, it will ever be as present as his strong personal attractiveness, the result at once of an easy and varied culture, and of a simple dignity of character, which impressed alike his family, his friends, and his pupils. His family consisted, at the time of his death, of his wife, four sons, and a daughter."

"The only mathematical book written by Chauvenet and not mentioned in the above sketch is a little book entitled 'Binomial Theorem and Logarithms,' published in 1843 for the use of midshipmen at the Naval School, Philadelphia."

As regards the quality of Professor Chauvenet's books, Prof. T. H. Safford, of Williams College, says: "This excellent man and lucid writer was admirably adapted to promote mathematical study in this country. His father, a Frenchman of much culture, trained him very thoroughly in the knowledge of the French language, even in its niceties. They habitually corresponded in that language; and the son was enabled to study the mathematical writings of his ancestral country in a way which enabled him to reproduce in English their ease and grace of style, as well as their matter. In these respects his works are far more attractive than those of ordinary English writers; his Trigonometry is much the best work on the subject which I know of in any language;his Spherical and Practical Astronomy is frequently quoted by eminent *continental* astronomers; and his Geometry has raised the standard of our ordinary text-books, of which it is by far the best existing."

Professor Chauvenet's books, especially his Geometry and Trigonometry, have been used in the best of American schools. Recently Professor Byerly, of Harvard University, brought out an *excellent* revised edition of the Geometry. In their originality, the works of Professor Chauvenet are admirably rigorous. The methods of investigation adopted in his Astronomy are in accordance with what may be called the modern school of practical astronomy—or more distinctly, the *German* school—at the head of which stands the unrivaled BESSEL. His Trigonometry and Astronomy are the first *American* works to introduce the consideration of *the general spherical triangle*, or that in which the six parts of the triangle are not subjected to the condition that they shall each be less than 180°, but may have any values less than 360°. Also, all ambiguity as to the species of the six parts of the triangle is removed by determining the parts, when necessary, by *two* of their trigonometric functions, usually the *sine* and the *cosine*. In adopting this admirable feature—mainly due to *Gauss*, Professor Chauvenet was years in advance of the English and *other* American astronomers. A *new* and *simple* demonstration of the formula for the prediction of the transits of the inferior planets over the sun's disc, he gives; while *Lagrange's* well-known formula in this connection, he renders *more accurate* by his introduction of a consideration with respect to the compression of the earth. Taking the fundamental formulae of Bessel's theory of eclipses, he deduces new and elegant solutions—and these *quite as exact* as the Besselian ones. In so far as the distinctive treatment of the *occultations of planets* by the moon, is concerned, Professor Chauvenet stands as the illustrious pioneer. His Trigonometry is still *the* book in the United States Naval Academy; and last session we had a special class from Annapolis, making up Naval-Academy shortcomings in Chauvenet's Plane and Spherical Trigonometry. In that mountainous county of Pike, in that wilderness-county of north-eastern Pennsylvania, in that venatorial elysium and piscatorial paradise of 'The Keystone State,' Professor William Chauvenet was born—think of it, readers of the MONTHLY. We acknowledge our indebtedness to *President* Regis Chauvenet, Colorado School of Mines, and to *Professor* Florian Cajori, of Colorado College, for material used in this biographical sketch.

THE INSCRIPTION OF REGULAR POLYGONS.

By LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago, Chicago, Illinois.

CHAPTER VI.

[Concluded from the January Number.]

IV. Let $n=mr$, where, n being odd, m and r are both odd. Then
$A_s-A_{m-s}-A_{m+s}+A_{2m-s}+A_{2m+s}-\ldots$

$$+(-1)^{\frac{r-1}{2}}A_{\frac{r-1}{2}m-s}+(-1)^{\frac{r-1}{2}}A_{\frac{r-1}{2}m+s}=0\ldots.(6).$$

Proof: $A_m-A_{2m}+A_{3m}-A_{4m}+\ldots.-(-1)^{\frac{r-1}{2}}A_{\frac{r-1}{2}m}=1$, being chords of the regular r-gon. Multiplying both sides by A_s, we find
$A_s=A_{m-s}+A_{m+s}-A_{2m-s}-A_{2m+s}+\ldots.$ By the method so often used we may prove that the r chords of (6) $A_s, -A_{m-s}, -A_{m+s}, A_{2m-s}$, etc., are the roots of $x^r-rx^{r-2}+\frac{r(r-3)}{1.2}x^{r-4}-\frac{r(r-4)(r-5)}{1.2.3}x^{r-6}+\ldots$

$$+(-1)^p\frac{r(r-p-1)(r-p-2)\ldots.(r-2p+1)}{1.2.3\ldots.p}x^{r-2p}+\ldots.\pm rx-A_{rs}=0\ldots.(7).$$

This may be proved directly from the trigonometric formula:

$$2\cos r\theta=2^r\cos^r\theta-2^{r-2}r\cos^{r-2}\theta+2^{r-4}\frac{r(r-3)}{1.2}\cos^{r-4}\theta$$

$$-\ldots.+(-1)^p2^{r-2p}\frac{r(r-p-1)(r-p-2)\ldots.(r-2p+1)}{1.2.3\ldots.p}\cos^{r-2p}\theta+\ldots.$$ from

which it follows that $A_s=2\cos\frac{s\pi}{mr}$ is a root of equation (7).

Also $-A_{m-s}=-2\cos\frac{(m-s)\pi}{mr}$ is a root of (7); for, θ then being $\frac{(m-s)\pi}{mr}$,

$2\cos r\theta=2\cos\frac{(m-s)\pi}{m}=-2\cos\frac{s\pi}{m}=-A_{rs}$. Similarly, $-A_{m+s}$, A_{2m-s}, A_{2m+s}, etc., are roots of (7). In another article I give a direct geometric-algebraic proof based upon the principles given above and the theory of symmetric functions.

If m is prime to r, one chord of each of the above groups (6) of r chords each is a root of our general equation (4) for a regular m-gon. For one and only one of the subscripts s, $m-s$, $m+s$, $2m-s$, $2m+s$, ... $\frac{r-1}{2}m-s$, $\frac{r-1}{2}m+s$ is always divisible by r, as is seen by writing them in the equivalent form: $rm-s$, $m-s$, $(r-1)m-s$, $2m-s$, $(r-2)m-s$, ... $\frac{r-1}{2}m-s$, $\frac{r+1}{2}m-s$, respectively. The remaining $r-1$ chords will be determined by equations whose degrees are given by the prime factors of $r-1$, — a chain of equations in which the coefficients of any one are linear functions of the roots of the preceding and of the roots of (4) for the m-gon. Hence, if the $\frac{m-1}{2}$ chords of the regular m-gon be found, we determine all the chords of the regular rm-gon by solving a series of equations whose degrees are the prime factors of $r-1$.

However, if m is divisible by r, the r chords in any of the above groups are all, or not one of them, roots of (4): for the subscripts s, $m-s$, $m+s$, $2m-s$, $2m+s$, ... are all or not one divisible by r, according as s is or is not divisible by r. Hence, by the grouping of the $\frac{n-1}{2}$ chords of the n-gon into $\frac{m-1}{2}$ groups of r chords each, we can not lower or avoid equations of the rth degree of the form (7). More definitely, if r be a prime number and if m be divisible by r, we must, for any grouping whatever of the chords of the rm-gon, solve one or more equations of degree r.

The regular polygon of mr sides depends for inscription, if m be prime to r, upon the same equations as does the regular m-gon, together with equations whose degrees are the prime factors of $r-1$; but, if m contains as factor the prime number r, upon an equation of degree r and of the form (7), in addition to those required by the regular m-gon.

To inscribe a regular polygon of $n=a^{\alpha}b^{\beta}c^{\gamma}$ .... sides, therefore, where a, b, c, ... are *different* prime numbers, it is necessary to solve $\alpha-1$ equations of degree a, $\beta-1$ of degree b, etc., besides equations whose degrees are given as the prime factors of $\frac{a-1}{2}$, $\frac{b-1}{2}$, $\frac{c-1}{2}$, ...

It follows that the regular $(2^x+1)m$-gon depends for inscription upon the same equations as the regular m-gon, provided 2^x+1 be a prime number, and is inscriptible if the latter is. Hence, a regular polygon of $(2^x+1)(2^y+1)(2^z+1)$.... sides, where the factors are *different* prime numbers, is geometrically inscriptible. We thus have Gauss' theorem:

A regular polygon the number of whose sides is a prime number of the form 2^x+1, or the product of two or more different primes of that form, or a power of 2 times such an expression, is geometrically inscriptible; and inversely.

Of the regular polygons with less than 200 sides, 31 are geometrically inscriptible:

3,4,5,6,8,10,12,15,16,17,20,24,30,32,34,40,48,51,60,64,68,80,85,96, 102, 120,128,136,160,170,192;

67 depend for inscription upon *cubics* only: 7,9,13,14,18,19,21,26,27, 28,35,36,37,38,39,42,45,52,54,56,57,63,65,70,72,73,74,76,78,81,84,90, 91, 95, 97, 104,105,108,109,111,112,114,117,119,126,130,133,135,140,144,146,148,152,153, 156,162,163,168,171,180,182,185,189,190,193,194,195;

23 depend upon *quintics* only: 11,22,25,33,41,44,50,55,66,75,82,88, 100,101,110,123,125,132,150,164,165,176,187; 17 depend upon *cubics* and *quintics*: 31,61,62,77,93,99,122,124,143,151,154,155,175,181,183,186,198;

8 depend on equations of 7th degree only: 29,58,87,113,116,145,174,197; 9 depend on equations of 3rd and 7th degrees: 43,49,86,98,127,129,147,172, 196; 2 on equations of 5th and 7th degrees: 71 and 142; the 40 remaining depend on equations of the 11th or higher degrees.

An idea of the comparative infrequency of the geometrically inscriptible regular polygons if we advance to large numbers is found in the fact (as shown by a complete table I have made) that there are only 206 of them with sides less than a million.

In a paper* on *The Number of Inscriptible Regular Polygons*, I proved that between the successive powers of 2 lie 1,2,3,4,5, etc. numbers giving inscriptible regular polygons; thus

[3],4,[5,6],8,[10,12,15],16,[17,20,24,30],32,[34,40,48,51,60],64,....

or generally, n such numbers lie between 2^n and 2^{n+1}, for every value of $n<32$. For $n \geqq 32$ but <128, 31 such numbers lie between 2^n and 2^{n+1}. For values of $n \geqq 128$, there would be 31 such numbers in each interval *if* (as is not yet known) $2^{128}+1$ is a composite number; but $31+l$ such numbers between 2^{128+l} and 2^{129+l}, if it is a prime number, l being <128.

Hence the number of inscriptible regular polygons below 2^x+1 sides, for $x<32$, is $\frac{1}{2}(x-1)(x+2)$; for $x \geqq 32$, but <128, is $(32x-497)$.

In conclusion, I will add a direct geometric proof of our fundamental theorem (5): $A_1-A_2+A_3-A_4+A_5-\dots-(-1)^pA_p=1$.

Suppose $A_1-A_2+A_3-\dots\pm A_p=x$. $\therefore$ $xA_1=A_1(A_1-A_2+A_3-\dots\pm A_{p-2}\pm A_{p-1}\pm A_p)=2+A_2-A_1-A_3+A_2+A_4-A_3-A_5+\dots\pm A_{p-3}\pm A_{p-1}\pm A_{p-2}\pm A_p\pm A_{p-1}\pm A_p=2+A_1-2(A_1-A_2+A_3-\dots\pm A_{p-2}\pm A_{p-1}\pm A_p)=2+A_1-2x$.

$\therefore$ $(x-1)(2+A_1)=0$. But $A_1+2=0$. $\therefore$ $x=1$.

Errata in October Number:—p.343, line 20 add *when n is a prime number*; p. 344, line 8 read $A_s^2=2-A_{n-2s}$.

* Bulletin of the New York Mathematical Society, February, 1894.

A PROPOSITION IN REFERENCE TO CENTRE OF GRAVITY, AND ITS DEMONSTRATION.

By J. W. NICHOLSON, A. M., LL.D., President and Professor of Mathematics, Louisiana State University and Agricultural and Mechanical College, Baton Rouge, Louisiana.

PROPOSITION. The point $P'(x',y',z')$ is the centre of gravity of the mass m if the sum (s) of the squares of the distances from P' to every point of m, is a minimum.

PROOF. Let $P(x,y,z)$ be any point of m, then the square of the distance PP' is $PP'^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$,

$$\text{and} \quad s = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}[(x-x')^2 + (y-y')^2 + (z-z')^2]\,dx\,dy\,dz.$$

Representing $\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}$ by $\int$ and $dx\,dy\,dz$ by dm,

$$\text{we have } s = \int[(x-x')^2 + (y-y')^2 + (z-z')^2]\,dm.$$

Since s is a minimum with respect to the independent variables.

x',y',z', we have $\frac{ds}{dx'} = 0$, $\frac{ds}{dy'} = 0$, and $\frac{ds}{dz'} = 0$; that is,

$$(1). \quad \int(x-x')\,dm = 0, \quad \therefore \quad x' = \frac{\int x\,dm}{\int dm};$$

$$(2). \quad \int(y-y')\,dm = 0, \quad \therefore \quad y' = \frac{\int x\,dm}{\int dm};$$

$$(3). \quad \int(z-z')\,dm = 0, \quad \therefore \quad z' = \frac{\int z\,dm}{\int dm}.$$

Q. E. D.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

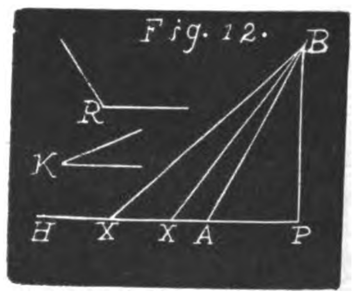
By GEORGE BRUCE HALSTED. A. M.. (Princeton); Ph. D., (Johns Hopkins), Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the January Number.]

SCHOLION II. In the three preceding theorems I have studiously set down this condition, that the cutting straight AP, or XA, is understood to be of a *designated length as great as you choose.*

For if, without any determinate extent of the cutting straight it be discussed precisely concerning the exhibiting and demonstrating of the concurrence of two straights at the apex of a certain triangle, whose angles at the base are given (less indeed than two right angles) as, suppose, one right, and the other less than a right by as much as two degrees, or, if you please, by less; who is so devoid of geometry that he could not immediately show the thing itself demonstratively?

For suppose (fig. 12) given any angle BAP, as, say, 88 degrees. If therefore from any point B of this AB, is let fall on the base AP (Eu. I. 12.) the perpendicular BP, it holds certainly that in this triangle ABP would be exhibited demonstratively the desired concurrence in this point B.



But if the other angle at the base is postulated, and it less than a right, as suppose 84 degrees, which indeed the given angle K represents: then (Eu. I. 23.) one would be able to make toward the parts of the straight AB an equal angle APD, PD meeting this AB in D some intermediate point of it. Wherefore the desired concourse is again obtained demonstratively in this point D.

But finally: if the other angle is postulated obtuse, but yet less than 92 degrees, lest with the other given angle BAP it should make up two rights: this may be represented in a certain angle R of 91 degrees. It is to be shown, that there is some one point X of this AP, to which the join BX makes an angle BXA equal to the given angle R of 91 degrees; so that therefore under a certain cutting straight AX the desired meeting in the point B many be obtained.

But we may proceed thus.

PA being produced to any point H, since the external angle BAH is (Eu. I. 13.) 92 degrees, since the interior angle BAP is by hypothesis 88 degrees; and again, (Eu. I. 16.) is greater not alone than the right angle BPA, but also, for the same reason, than any obtuse angle BXA, the point X being assumed wherever you choose within this PA, and indeed always growing greater as the point X is assumed nearer to the point A, (Eu. I. 16.) : it is an evident consequence, that between those angles, one of 90 degrees at the point P, and the other of 92 degrees in the point A, one angle BXA is found, which is 91 degrees, truly equal to the given angle R. None the less, omitting this last observation about the obtuse angle, it is necessary most diligently to take care that the difficulty of this proposition [axiom] of Euclid be fixed in this, that it asserts the meeting of two straights; especially in that part in which they make with the cutting straight two angles less than two right angles; and assuredly that it asserts the aforesaid meeting thus, *of whatever length be the assigned transversal.*

For otherwise (as I have already mentioned in the preceding scholion) I will demonstrate that general meeting solely from the admitted meeting of this sort, when one of the angles is right; and indeed, even if it be admitted not for any assignable finite transversal, but alone admitted within the limits of any assigned very small transversal.

[To be continued.]

ARITHMETIC.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

36. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

It costs $C=\$22$ to paper a room $a=18$ feet long, $b=15$ feet wide, and $c=10$ feet high, with paper (m/n)th, $=\frac{3}{4}$, of a yard wide. Find the price of the paper per roll of $R=12$ linear yards.

Solution by the PROPOSER.

Making no allowance for "matching", the number of linear yards of paper required is $Y=\frac{1}{3}[2(a+b)c+ab](n/m), =413\frac{1}{3}$; and, consequently, the number of rolls of paper required is $N=Y/R=34\frac{4}{9}$.

Hence the price of the paper per roll is

$$P=\frac{3(m/n)R}{2(a+b)c+ab} \text{ of } \$C, =\$\tfrac{99}{155}=63\tfrac{7}{8} \text{ cents.}$$

Professor J. F. W. Scheffer, P. S. Berg, and J. A. Calderhead get \$2.70 as the result. Professor John Faught and I. L. Beverage get \$1 91$\frac{3}{7}$, and Professor T. W. Palmer gets \$178$\frac{3}{7}$. These different results are due to different interpretations of the problem.

37. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

I have three jars, A, B, and C, holding respectively $a=1$, $b=3$, and $c=5$ gallons. A is empty, B is full of water, and C is full of wine. I *fill* A from B; then I *fill* up B from C and pour the contents of A into C. After *repeating* this operation how much wine is there in B? How much in C?

I. Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

After the first operation, B contains 1 gallon of wine, C contains 4 gallons. In the second operation $\frac{4}{5}$ gallons of wine are drawn from C and $\frac{1}{3}$ gallons added to C. C contains, $4-\frac{4}{5}+\frac{1}{3}=3\frac{8}{15}$ gallons of wine.
B contains, $5-3\frac{8}{15}=1\frac{7}{15}$ gallons of wine.

II. Solution by Professor T. W. PALMER, University of Alabama, I. L. BEVERAGE, Monterey, Virginia, and the PROPOSER.

After the *first* operation, A is empty, B contains $(b-a)$ gallons of water and a gallons of wine, and C contains $(c-a)$ gallons of wine and a gallons of water. In so far as changes in the proportional parts of the wine are effected by the *repetition*, A is empty, B has gained from C $[(c-a)\div c]$ of a gallons, and C has gained from B *through* A $(a\div b)$ of a gallons. Hence, after the repetition, the number of gallons of wine in B is

$$G_B=\left[1+\frac{c-a}{c}-\frac{a}{b}\right] \text{ of } a=\left[2-\left(\frac{1}{b}+\frac{1}{c}\right)a\right] \text{ of } a, \ =1\tfrac{7}{15}.$$

Consequently the number of gallons of wine in C is

$$G_C=c-G_B=c-\left[2-\left(\frac{1}{b}+\frac{1}{c}\right)a\right] \text{ of } a, \ =3\tfrac{8}{15}.$$

III. Solution by Professor J. F. W. SCHEFFER, Hagerstown, Maryland.

Let, after x such operations as described, the quantity of wine in the vessel B be $F(x)$, then $b-F(x)$ will be the quantity of water in B, $c-F(x)$ the quantity of wine in C and $F(x)$ the quantity of water in C.

Repeating the operation once more, we have in vessel B,

$$\frac{b-a}{b}F(x) \text{ wine} + \frac{b-a}{b}[b-F(x)] \text{ water} + \frac{a}{c}[c-F(x)] \text{ wine} + \frac{a}{c}F(x) \text{ water};$$

and in C, $\frac{c-a}{c}[c-F(x)]$ wine $+\frac{c-a}{c}F(x)$ water $+\frac{a}{b}F(x)$ wine $+\frac{b-F(x)}{b}$ water. $\therefore \frac{b-a}{b}F(x)+\frac{a}{c}[c-F(x)]=F(x+1)$ an equation in Finite Differences. Resolving, we find $F(x)=\frac{bc}{b+c}+C\left(\frac{bc-ab-ac}{bc}\right)^x$.

For $x=0$, $F(x)=0$. $\therefore C=-\frac{bc}{b+c}$; $\therefore F(x)=\frac{bc}{b+c}\left[1-\left(\frac{bc-ab-ac}{bc}\right)^x\right]$.

The quantity of wine in C is, of course, $=c-F(x)$.

For $x=2$, we have $F(x)=\frac{a(2bc-ab-ac)}{bc}$. For the numerical values we have $F(x)=\frac{15}{8}[1-(\frac{7}{15})^x]$ and $F(2)=1\frac{7}{15}$. The value to which $F(x)$ approximates is $\frac{bc}{b+c}$.

38. Proposed by J. A. CALDERHEAD, B. Sc., Superintendent of Schools, Limaville, Ohio.

What must be the thickness of a 36-inch shell, in order that it may weigh 1 ton; supposing a 13-inch shell to weigh 200 pounds, when two inches thick?

I. Solution by EDWARD R. ROBBINS, Princeton, New Jersey.

Given shell....Vol. $=\frac{734\pi}{3}$. Required shell of thickness $=x$....

Volume $=\frac{(4x^3-72x^2+1296x)\pi}{3}$. Now.... $\frac{734\pi}{3}:\frac{4(x^3-18x^2+324x)\pi}{3}::$ 1:10 which gives.... $x^3-18x^2+324x-1835=0$, whence $x=7.4817$ in. the required thickness.

II. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia, and J. W. WATSON, Middle Creek, Ohio.

Volume 13-inch sphere $=\frac{1}{6}\pi 13^3=\frac{2197\pi}{6}$ cubic inches.

Volume of hollow $=\frac{1}{6}\pi 9^3=\frac{243\pi}{2}$ cubic inches.

Volume of 36-inche sphere $=\frac{1}{6}\pi 36^3=7776\pi$ cubic inches.

Volume of hollow of 36-inch sphere $=\frac{1}{6}\pi D^3$ cubic inches.

$\frac{2197\pi}{6}-\frac{243\pi}{2}=\frac{734\pi}{3}$ cubic inches = volume of 13-inch shell.

$\therefore$ $\frac{734\pi}{3}$ cubic inches weigh 200 pounds.

$\therefore$ 1 cubic inch weighs $\frac{600}{734\pi}=\frac{300}{367\pi}$ pounds.

7776π cubic inches weigh $7776\pi\times\frac{300}{367\pi}=\frac{2332800}{367}$ pounds.

$\frac{1}{6}\pi D^3$ cubic inches weigh $\frac{1}{6}\pi D^3\times\frac{300}{367\pi}=\frac{50D^3}{367}$ pounds.

But $\frac{2332800}{367}-\frac{50D^3}{367}=2000$. $\therefore$ $D^3=31976$. $\therefore$ $D=31.74$ inches.

$\therefore$ $36-31.74=4.26$, $4.26\div 2=2.13$ inches.

$\therefore$ 2.13 inches is the thickness of the 36-inch shell.

PROBLEMS.

44. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

A, B, and C together bought a ship. A paid for the a/bth, $=\frac{2}{5}$th, part of the ship. B paid for the m/nth, $=\frac{1}{3}$th, part of the ship. C paid $\$M$, $=\$2000$. How many dollars did A, and B, pay?

45. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

In running a mile, A can give B $a=20$ yards; B can give C $b=88$ yards. How many yards can A give C?

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

34. Proposed by ROBERT J. ALEY, A. M., Professor of Mathematics, Indiana University, Bloomington, Indiana.

$$\Sigma_1^n \frac{(n+2)^2}{n(n+4)}=\text{what?}$$

Solution by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

$$\Sigma_1^n \frac{(n+2)^2}{n(n+4)}=\Sigma_1^n\left[1+\frac{4}{n(n+4)}\right]=\Sigma_1^n(1)+4\Sigma_1^n\frac{1}{n(n+4)}=n+4\Sigma_1^n\frac{1}{n(n+4)}.$$

$$\text{But } 4\Sigma_1^n\frac{1}{n(n+4)}=\left(\frac{1}{1}-\frac{1}{5}\right)+\left(\frac{1}{2}-\frac{1}{6}\right)+\left(\frac{1}{3}-\frac{1}{7}\right)+\left(\frac{1}{4}-\frac{1}{8}\right)+\dots$$

$$+\left(\frac{1}{n-4}-\frac{1}{n}\right)+\left(\frac{1}{n-3}-\frac{1}{n+1}\right)+\left(\frac{1}{n-2}-\frac{1}{n+2}\right)$$

$$+\left(\frac{1}{n-1}-\frac{1}{n+3}\right)+\left(\frac{1}{n}-\frac{1}{n+4}\right)=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{n+1}-\frac{1}{n+2}$$

$$-\frac{1}{n+3}-\frac{1}{n+4}=\left(1-\frac{1}{n+1}\right)+\left(\frac{1}{2}-\frac{1}{n+2}\right)+\left(\frac{1}{3}-\frac{1}{n+3}\right)+\left(\frac{1}{4}-\frac{1}{n+4}\right)$$

$$=\frac{n}{n+1}+\frac{n}{2(n+2)}+\frac{n}{3(n+3)}+\frac{n}{4(n+4)}.$$

$$\therefore\quad \Sigma_1^n \frac{(n+2)^2}{n(n+4)} = n\left[1+\frac{1}{n+1}+\frac{1}{2(n+2)}+\frac{1}{3(n+3)}+\frac{1}{4(n+4)}\right].$$

Also solved by *Professor G. B. M. Zerr.*

35. **Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee. Knoxville, Tennessee.**

Prove that the product of two numbers, each the sum of four (4) squares may be expressed as the sum of four squares in 48 different ways and unite some or all of the 48 ways.

Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let $(a^2+b^2+c^2+d^2)(e^2+f^2+g^2+h^2)=A$ = the product of the two numbers. From Euler's Theorem we get

$$A=(ae+bf-cg-dh)^2+(af-be-ch+dg)^2$$
$$+(-ag+bh-ce+df)^2+(ah+bg+cf+de)^2.$$

$$=(-ae+bf-cg+dh)^2+(af+be+ch+dg)^2$$
$$+(ag+bh-ce-df)^2+(ah-bg-cf+de)^2.$$

$$=(ae-bf-cg+dh)^2+(af+be-ch-dg)^2$$
$$+(ag+bh+ce+df)^2+(-ah+bg-cf+de)^2.$$

$$=(ae+bf+cg+dh)^2+(-af+be-ch+dg)^2$$
$$+(ag-bh-ce+df)^2+(ah+bg-cf-de)^2.$$

$$=(ae-bf+cg+dh)^2+(af+be-ch+dg)^2$$
$$+(ag-bh-ce-df)^2+(ah+bg+cf-de)^2.$$

$$=(ae-bf-cg-dh)^2+(af+be+ch-dg)^2$$
$$+(ag-bh+ce+df)^2+(ah+bg-cf+de)^2.$$

$$=(ae+bf-cg+dh)^2+(af-be+ch+dg)^2$$
$$+(ag+bh+ce-df)^2+(ah-bg-cf-de)^2.$$

$$=(ae+bf+cg-dh)^2+(af-be-ch-dg)^2$$
$$+(ag+bh-ce+df)^2+(ah-bg+cf+de)^2.$$

The sum of four squares in eight different ways by combination of signs.

$$A=(ae+bf-cg-dh)^2+(af-be-ch+dg)^2$$
$$+(-ag+bh-ce+df)^2+(ah+bg+cf+de)^2.$$

$$=(ag+bf-ch-de)^2+(af-bg-ce+dh)^2$$
$$+(-ah+be-cg+df)^2+(ae+bh+cf+dg)^2,$$

$$=(ah+bf-ce-dg)^2+(af-bh-cg+de)^2$$
$$+(-ae+bg-ch+df)^2+(ag+be+cf+dh)^2.$$
$$=(ae+bg-cf-dh)^2+(ag-be-ch+df)^2$$
$$+(-af+bh-ce+dg)^2+(ah+bf+cg+de)^2.$$
$$=(af+bg-ch-de)^2+(ag-bf-ce+dh)^2$$
$$+(-ah+be-cf+dg)^2+(ae+bh+cg+df)^2.$$
$$=(ah+bg-ce-df)^2+(ag-bh-cf+de)^2$$
$$+(-af+be-cg+dh)^2+(ae+bf+ch+dg)^2.$$

the sum of four squares in six different ways by combination of letters.

Since the signs of each of these six can form the sum of four squares in eight different ways, the whole number of ways is $8\times6=48$ different ways.

PROBLEMS.

46. Proposed by Professor WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics in the Ohio University, Athens, Ohio.

Find θ from $\cos\theta+\cos 3\theta+\cos 5\theta=0$.

47. Proposed by LEONARD E. DICKSON, A. M., Fellow in Mathematics, University of Chicago, Chicago, Illinois.

Prove that $(-1)(-1)=+1$.

GEOMETRY.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

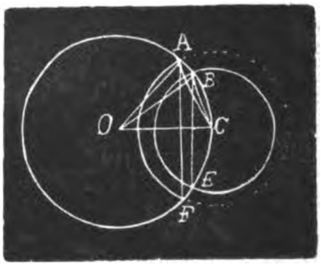
SOLUTIONS OF PROBLEMS.

34. Proposed by T. JOHN COLE, Columbus, Ohio.

A circular field contains 10 acres. A horse is tied to the fence with a rope sufficiently long to graze over one acre. Find length of the rope (1) when the horse is on the inside (2) when he is on the outside of the fence.

Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let C be the point to which the rope is fastened, A and D two points in the circumference to which the horse can graze when on the inside, B and E the points in the circumference to which the horse can graze when on the outside. Let O be the center of the given circle.



Let $OA=a=\dfrac{40}{\sqrt{\pi}}$, the radius of the given circle, and $\angle ACO=\theta$, $\angle BCO=\phi$. Then we have $AC=2a\cos\theta$, $BC=2a\cos\phi$. The area common to the two circles in the first case $=a^2(\pi+2\theta\cos2\theta-\sin2\theta)$. The area common to the two circles in the second case $=a^2(\pi+2\phi\cos2\phi-\sin2\phi)$.

Therefore the area upon which the animal can graze upon the inside of the circle is

$$a^2(\pi+2\theta\cos2\theta-\sin2\theta)=\tfrac{1}{10}\pi a^2 \ldots\ldots(1).$$

The area upon the outside is

$$4\pi a^2\cos^2\phi-a^2(\pi+2\phi\cos2\phi-\sin2\phi)=\tfrac{1}{10}\pi a^2 \ldots\ldots(2).$$

From (1) $9\pi2+0\theta\cos2\theta-10\sin2\theta=0$. $9\pi+20\theta\cos2\theta-10\sin2\theta=0$.

From (2) $40\pi\cos^2\phi-20\phi\cos2\phi+10\sin2\phi=11\pi$.

Solving by the method of double position,

$\theta=76^\circ\ 21'\ 44''.04$,

$\phi=77^\circ\ 38'\ 25''$.

$AC=2a\cos\theta=10.64216$ rods,

$BC=2a\cos\phi=9.65892$ rods.

Good solutions to this problem were received from *J. F. W. Scheffer*, and *P. S. Berg*.

35. Proposed by LEONARD E. DICKSON, M. A., Fellow in Mathematics, The University of Chicago.

Determine the equation of lowest degree (cubic) upon which depends the inscription of the regular polygon of 37 sides.

Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Using the proposers notation as given in his excellent papers in the MONTHLY, we get for $n=37$, the following order of subscripts: 1,2,4,8,16,5,10,17,3,6,12,13,11,15,7,14,9,18.

Hence the groups are $(A_1-A_8-A_{10}-A_6+A_{11}-A_{14})=A$, $(-A_2-A_{16}+A_{17}-A_{12}+A_{15}+A_9)=B$, $(-A_4+A_5+A_3+A_{13}+A_7-A_{18})=C$.

$A+B+C=1$. $AB=5(-A_1-A_8-A_{10}-A_6+A_{11}-A_{14})$
$-4(-A_2-A_{16}-A_{17}-A_{12}-A_{15}+A_9)$
$-3(-A_4+A_5+A_3+A_{13}+A_7-A_{18})$.

$\therefore\ AB=-(5A+4B+3C)=-5+B+2C$.

By symmetry, $AC=-5+A+2B$, $BC=-5+C+2A$.

$\therefore\ AB+AC+BC=-15+3(A+B+C)=-12$.

$ABC=-A(5-C-2A)=-A(3+2B+C)=-(3A+2AB+AC)$.

$\therefore\ ABC=-(3A-10+2B+4C-5+A+2B)=-4(A+B+C)+15$.

$\therefore\ ABC=-11$.

$\therefore\ A,B,C$ are the roots of the equation $x^3-x^2+12x-11=0$, which is the equation required.

38. Proposed by O. W. ANTHONY, Mexico, Missouri.

From two points, one on each of the opposite sides of a parallelogram, lines are drawn to the opposite vertics. Through the points of intersection of these lines a line is drawn. Prove that it divides the parallelogram into two equal parts.

Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics in the University of Mississippi, University P. O., Mississippi.

Let $OABC$ be the parallelogram, O the lower left-hand vertex, OA the base, D and E points on OA and CB respectively.

If $OA=b$, $OC=a$, $OD=n$, $CE=m$, then, taking OA as the X-axis and OC as the Y-axis, the points O,A,B,C,D, and E will be given by the co-ordinates $(0,0)$, $(b,0)$, $b,a)$, $(0,a)$, $(n,0)$, and (m,a) respectively.

It follows that the equation of OE is $\frac{y}{a}=\frac{x}{m}$: . . . . (1).

The equation of CD is $\frac{y}{a}+\frac{x}{n}=1$; . . . . (2).

The equation of BD is $y=\frac{-a}{n-b}(x-n)$. . . . . (3).

The equation of EA is $y=\frac{-a}{b-m}(x-b)$. . . . . . (4).

From (1) and (2) the intersection of OE and CD is

$$\left(\frac{mn}{m+n}, \frac{an}{m+n}\right)$$ which denote by (x',y').

From (3) and (4) the intersection of BD and EA is

$$\left(\frac{b^2-mn}{2b-m-n}, \frac{-a(n-b)}{2b-m-n}\right)$$ which denote by (x'',y'').

The equation of the line passing through these points of intersection is

$y-y'=\frac{y'-y''}{x'-x''}(x-x')$. . . . (5).

If the center of the parallelogram is on this line its co-ordinates, $\left(\frac{b}{2}, \frac{a}{2}\right)$, will satisfy (5).

Substituting and reducing, $\frac{1}{2}-\frac{n}{m+n}=\frac{m-n}{2(m+n)}$, or (5) is satisfied.

Since every line which passses through the center of aparallelogram divides it into two equal parts, the proposition is established.

PROBLEMS.

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39. Proposed by J. K. ELLWOOD, Principal of Colfax Schools, Pittsburg, Pennsylvania.

If on the three sides of any plane triangle equi'ateral triangles be described, the lines joining the centres of these equilateral triangles form an equilateral triangle.

40. Proposed by J. C. CORBIN, Pine Bluff, Arkansas.

If R, r, r_1, r_2, and r_3 be, respectively, the radii of the circumscribed, inscribed, and escribed circles of a $\triangle$, prove $r_1+r_2+r_3-r=4R$.

41. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

Find the length (x) of a rectangular parallelopiped $b=5$ ft. and $h=3$ ft., which can be *diagonally inscribed* in a similar parallelopiped $L=83$ ft., $B=64$ ft., and $H=50$ ft.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

26. **Proposed by Professor J. F. W. SCHEFFER, M. A., Hagerstown, Maryland.**

According to Bessel, the ratio of the squares of the polar diameter of the earth to that of the equatorial diameter, is .9933254. Find what *latitude* the angle made by a body falling to the earth, with a perpendicular to the surface, is greatest. Find, also, this maximum angle.

Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Let ϕ=the required geographical latitude, and ϕ'=the geocentric latitude of the same place; then *Chauvenet's Spherical and Practical Astronomy*, Vol. I., p. 98, we deduce $\phi'=\tan^{-1}[(b^2/a^2)\tan\phi]=\tan^{-1}[(1-e^2)\tan\phi]$.

$\therefore\ (\phi-\phi')=\phi-\tan^{-1}[(1-e^2)\tan\phi]$, =a Maximum.

$$\therefore\ \frac{d(\phi-\phi')}{d\phi}=1-\frac{(1-e^2)(1+\tan^2\phi)}{1+(1+e^2)^2\tan^2\phi}=0.$$

$$\therefore\ \phi=\tan^{-1}\left[\sqrt{\left(\frac{1}{1-e^2}\right)}\right]=\tan^{-1}(1.0033541)=45^\circ\ 5'\ 45''.32,$$

and $\phi'=\tan^{-1}(.9966571)=44^\circ\ 54'\ 14''.67$.

Hence $(\phi-\phi')=11'\ 30''.65$; and this result is found in the already-named Manual of Astronomy, Vol. II., third Table, p. 577.

II. **Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.**

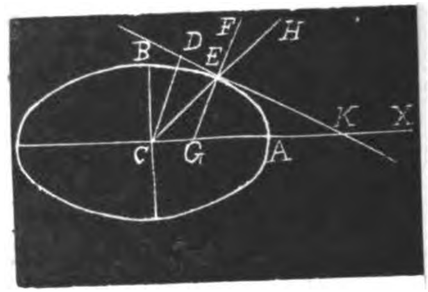
Not taking into account the eastward deviation due to the rotation of the earth we can proceed as follows:

Let HEC be the direction the body falls, FEG the perpendicular to the earth's surface at E, DEK the tangent to the meridian at E, $CA=a$,

$CB=b$, $\angle ECA=\theta$, $\angle DCE=\angle CEG=\phi$, $\angle EKX=\beta$, co-ordinates of $E=(x,y)$.

Then $\tan\beta=-\dfrac{b^2x}{a^2y}$, $\tan\theta=\dfrac{y}{x}$, also $\beta=90^\circ+\theta+\phi$, $\tan\beta=\tan(90^\circ+\theta+\phi)$

$$=-\cot(\theta+\phi).$$



$$\therefore\ \frac{b^2x}{a^2y}=\cot(\theta+\phi)=\frac{\cot\theta\cot\phi-1}{\cot\theta+\cot\phi}$$

$$=\frac{\frac{x}{y}\cot\phi-1}{\frac{x}{y}+\cot\phi}.$$

$\therefore\ \tan\phi=\dfrac{a^2-b^2}{a^2b^2}xy=\text{maximum}\ldots.(1).\quad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\ldots.(2).$

The first differentials of (1) and (2) give $b^2x^2=a^2y^2$.

$\therefore\ x=\dfrac{a}{\sqrt{2}},\ y=\dfrac{b}{\sqrt{2}}.\ \therefore\ \tan\theta=\dfrac{b}{a}=(.9933254)^{\frac{1}{2}}=.996659.\ \therefore\ \theta=44^\circ\ 54'$ $14''.9=$ the latitude. $\tan\phi=\dfrac{a^2-b^2}{2ab}=\dfrac{.0066746}{1.993318}=.003348.\ \therefore\ \phi=1'\ 30''.5$ $=$ maximum angle made with the perpendicular.

Also solved by Professor *C. W. M. Black*, and the *Proposer*.

PROBLEMS.

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36. Proposed by H. C. WHITAKER, B. Sc., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

A cube is revolved on its diagonal as an axis. Define the figure described and calculate its volume.

37. Proposed by J. A. CALDERHEAD, Superintendent of Schools, Limaville, Ohio.

A man ties two mules—one to the outside of a circular wall, the other to the inside. If the lengths of the ropes of each is one-fourth the circumference of the wall, and both together can graze over one acre of ground: find the circumference of the wall.

MECHANICS.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

14. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University Post Office.

"The center of a sphere of radius c moves in a circle of radius a and generates thereby a solid ring, as an anchor ring: prove that the moment of inertia of this ring about an axis passing through the center of the direct circle and perpendicular to a plane is $\frac{\pi^2\rho ac^2}{4}(4a^2+3c^2)$."

IV. Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let E be the centre of the sphere, AB, the axis of revolution, be the axis of x, F any point in area of the circle, $DF=y$, $HF=y'$, $CE=a$, $GE=c$.

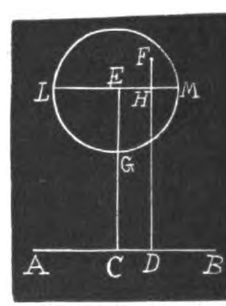
Let dA be the element of area, then the volume of the elementary ring generated by dA is $2\pi y dA$, and its mass, $2\pi\rho y dA$.

$\therefore$ the moment of inertia of this elementary ring relative to the axis of x, is $2\pi\rho y^3 dA$.

$\therefore$ the moment of inertia required $=I=2\pi\rho\Sigma y^3 dA$.

$I=2\pi\rho\Sigma(a+y')^3 dA$, since $y=a+y'$,

$=2\pi\rho\Sigma(a^3+3a^2y'+3ay'^2+y'^3)dA$.



But the curve is symmetrical with respect to the axis LM. $\therefore$ $\Sigma y'dA=0$, $\Sigma y'^3dA=0$, and by definition, $\Sigma y'^2dA=Ak^2=\pi c^2k^2$. $\therefore$ $I=2\pi\rho a\times\pi c^2(a^2+3k^2)$

$=2\pi^2\rho ac^2(a^2+3k^2)$; but $k^2=$ radius of gyration $=\frac{c^2}{4}$.

$\therefore$ $I=2\pi^2\rho ac^2(a^2+\frac{3}{4}c^2)=\frac{\pi^2\rho ac^2}{2}(4a^2+3c^2)$.

V. Solution by P. H. PHILBRICK, M. Sc., C. E., Lake Charles, Louisiana.

$OP=a=$ the radius of directing circle and $Pa=c=$ radius of the sphere.

Take tangent Px for the axis of x. Let $aPx=\theta$, $aPc=d\theta$. Draw abR and cdN perpendicular to Px. Take m at middle of ab. Now $am=c\cos\theta$, $\therefore$ $aR=a+c\cos\theta$ and $bR=a-c\cos\theta$. Thickness of $abcd=c\cos\theta d\theta$.

Area of circle, radius aR, $=\pi(a+c\cos\theta)^2$.

∴ vol. of circular lamina $aR=\pi(a+c\cos\theta)^2\, c\cos\theta d\theta$ and moment of Inertia about axis through O perpendicular to plane

$$OPx=I=\frac{\pi}{2}(a+c\cos\theta)^4\, c\cos\theta d\theta.$$

Similarly, moment of Inertia of lamina $bR=I'$

$$=\frac{\pi}{2}(a-c\cos\theta)^4\, c\cos\theta d\theta.$$

∴ moment of Inertia of lamina $ab=I-I'=I''=4\pi ac^2(a^2\cos^2\theta d\theta+c^2\cos^4\theta d\theta)$.

$$\text{But}\int_0^\pi \cos^2\theta d\theta=\frac{\pi}{2};\ \text{and}\int_0^\pi \cos^4\theta d\theta=\tfrac{3}{4}\int_0^\pi \cos^2\theta d\theta=\frac{3\pi}{8}.$$

∴ $I''=\frac{\pi^2ac^2}{4}(8a^2+6c^2)$. The moment of Inertia about an axis through O and in the plane OPx is of course one half of the above amount, or $\frac{\pi^2ac^2}{4}(4a^2+3c^2)$.

[NOTE.—An excellent solution of this problem was received from Professor Wiggins, Richmond, Indiana, but as it has been lost we are unable to publish it.—Editor.]

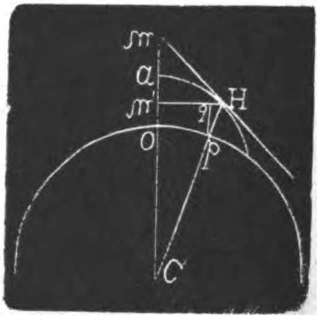
15. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

Show that the *eastward deviation* of bodies falling from a great height is

$$E_d=\frac{4\pi t(H-\frac{1}{2}\triangle)\cos\phi}{3T}.$$

Solution by the PROPOSER.

Let r=the radius of the earth, T=86164.2=the *mean-time* seconds in a siderial day, ϕ=the observer's latitude, $(r+H)$=the distance from the center of the earth to the point from which the body fell, and t=the time of the body's motion in seconds; then the *horizontal velocity* of the falling body in the direction of the tangent to the circle of latitude at the place of observation, is



$V=\frac{2\pi(r+H)\cos\phi}{T}$ ....(1): and this velocity is caused by the rotation of the earth. Obviously the *horizontal space* described by the body in the time t, is $S_1=Vt=\frac{2\pi t(r+H)\cos\phi}{T}$ ....(2).

The point on the earth and vertically below the point from which the body fell, passes during this time t through an arc

$$S_2=\frac{2\pi tr\cos\phi}{T}\ ....(3).$$

Hence, the eastward deviation of the falling body, if *the direction of gravity remained parallel toward this body*, would become

$E_a = S_1 - S_2 = \frac{2\pi t H\cos\phi}{T}$ ....(4); although this *change of direction* is not great, yet it must not by any means be neglected—even though the variation in the *intensity* of gravity is frequently neglected. Suppose the body in its descent to have arrived at H, and let $m'H = x$. Let w represent the body's velocity in the direction $m'H$, and v that in the direction HC; then $dv = g dt$ ....(5).

Considering $HC = r$, we have the proportion,

$$-dw : dv :: Hq : Hp :: m'H : HC :: x : r.$$

$\therefore \quad dw = -\frac{x}{r} dv, = -\frac{gx}{r} dt$ ....(6), the negative sign showing that w decreases as x increases. Since $w = dx/dt$, (6) becomes

$$dw = -\frac{gx dx}{rw}. \quad \therefore \quad w dw = -\frac{g}{r} x dx \text{ ....(7).}$$

Integrating (7) and remembering that when $x = 0$ the initial value of w is V as given in (1), we have $w^2 = V^2 - \frac{gx^2}{r}$. $\therefore$ $w = V\sqrt{\left(1 - \frac{gx^2}{rV^2}\right)}$ ....(8).

With $w = dx/dt$, we now easily deduce

$$V dt = \frac{dx}{\sqrt{\left(1 - \frac{gx^2}{rV^2}\right)}} = \left(1 + \frac{1}{2} \cdot \frac{gx^2}{rV^2} + \text{higher powers of } x \text{, which may be neglected.}\right) dx \text{ ....(9).}$$

Integrating (9), observing that $C = 0$ since $x = 0$ when $t = 0$, and without *appreciable* error putting $x^3 = (Vt)^3$, we have the expression

$$x = Vt - \frac{1}{6} \cdot \frac{gVt^3}{r} \text{ ....(10).}$$

If $\triangle$ represents the *excess of descent* in vacuo above that in air, we have

$$\frac{1}{2} g t^2 = H + \triangle \text{ ....(11).} \quad \therefore \quad x = Vt - \frac{(H+\triangle)Vt}{3r} \text{ ...(12).}$$

Transforming (12) by means of (1) and without *sensible* error writing *unity* for $(r+H)/r$, we have $x = \frac{2\pi t[(r+H) - \frac{1}{3}(H+\triangle)]\cos\phi}{T}$ ....(13).

For obvious reasons subtracting (3) from (13), etc., we obtain

$$E_a = \frac{4\pi t(H - \frac{1}{2}\triangle)\cos\phi}{3T} \text{ ....(14).}$$

which is the *expression* for the eastward deviation of bodies falling from a great height, given in *Young's General Astronomy*.

Also solved by *G. B. M. Zerr*. His solution will appear next month.

PROBLEMS.

22. Proposed by DE VOLSON WOOD, C. E., Professor of Mechanical and Electrical Engineering in Stevens Institute of Technology, Hoboken, New Jersey.

A prismatic bar having a uniform angular velocity ω and a linear velocity of v feet per second, suddenly snaps (by the disappearance of the cohesive force) into an indefinite number of equal parts; required the resultant angular velocity of each piece and the locus of the parts at the end of t seconds after rupture.

23. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University Post Office., Mississippi.

A heavy particle is placed upon the surface $\sqrt{x}+\sqrt{y}+\sqrt{z}=c$. The axis of z being vertical and the coefficient of friction being $1/\sqrt{2}$, show that a point of equilibrium (all friction possible being brought into action) z is a harmonical mean between x and y.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

13. Proposed by CHARLES E. MYERS, Canton, Ohio.

A soap bubble 2 inches in diameter, is filled with one part of hydrogen gas and 15 parts of air. If the bubble just floats in the air, find the thickness of the film.

I. Solution by H. C. WHITAKER, B. S., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

Take, as the unit of weight, the weight of air filling a sphere of one inch radius, then the weight of a corresponding volume of hydrogen is .06927 and of a corresponding volume of water is 792.24. Take x=inner radius of sphere. The weight of the hydrogen is $\frac{.06927}{16}x^3$;

The weight of the air in the bubble is $\frac{15}{16}x^3$;

The weight of the water is $792.24(1-x^3)$;

The weight of the air displaced is 1.

Making the sum of the first three of these equal to the last and solving, x=.999977, and the required thickness is .000023 inches.

II. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

In solving this problem many things should be considered. We should know the temperature, the barometric pressure and the dew point in order to calculate the amount of aqueous vapor present in the air.

Not knowing the above it will be well to proceed as follows:

Let r=outside radius of bubble, x=inside radius.

ρ=density of air, ρ_1=density of hydrogen, ρ_2=density of soap film, — all at the given temperature and pressure and all referred to normal air as standard.

$$\text{Then } \tfrac{4}{3}\pi(r^3-x^3)=\text{volume of soap film},$$

$$\tfrac{4}{3}\pi\times\frac{x^3}{16}=\text{volume of hydrogen},$$

$$\tfrac{4}{3}\pi\times\frac{15x^3}{16}=\text{volume of air}.$$

$$\therefore\ \tfrac{4}{3}\pi(r^3-x^3)\rho_2+-\tfrac{4}{3}\pi\times\frac{x^3}{16}\rho_1+\tfrac{4}{3}\pi\times\frac{15x^3}{16}\rho=\tfrac{4}{3}\pi r^3\rho,$$

$$\therefore\ 16r^3\rho_2-16x^3\rho_2+x^3\rho_1+15c^3\rho=16r^3\rho,$$

$$\therefore\ x^3=\frac{16r^3(\rho_2-\rho)}{16\rho_2-15\rho-\rho_1},\quad\therefore\ x=2r\sqrt[3]{\left(\frac{2(\rho_2-\rho)}{16\rho_2-15\rho-\rho_1}\right)}.$$

$$r-x=r-2r\sqrt{\left(\frac{2(\rho_2-\rho)}{16\rho_2-15\rho-\rho_1}\right)}=\text{required thickness}.$$

Let the conditions be normal and suppose ρ_2, referred to water, is 1.1. Then since air, referred to water, is .001293, ρ_2, referred to air, is 850.73473, $\rho=1$, $\rho_1=.0693$, $r=1$.

Then $r-x=1-2\sqrt[3]{\left(\frac{1199.46946}{13596.68638}\right)}=.000023$.

Also solved by *P. S. Berg*, *F. P. Matz*, and the *Proposer*.

14. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

I have a glass paper-weight in the form of a regular icosahedron. I let the sun's rays fall upon it, at various angles, also upon one of the vertices. How many complete spectra will be formed? How many will be of white light? What position will give maximum number of spectra?

[No solution to this problem has as yet been furnished by our contributors, and I see no way of solving it. If a solution is possible it will be a very pretty one. —EDITOR.]

PROBLEMS.

22. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

From what kind of dry wood must a ship's log be cut, in order that the log may float with its center of gravity at the water's surface?

PROBLEMS.

23. Proposed by G. B. M. ZERR, A. M., Principal High School, Staunton, Virginia.

Pliny says, "Thales determined the cosmical setting of the Pleiades to have happened in his time 25 days after the vernal equinox." Determine the time when Thales lived from the following data:—Latitude of Miletus 37° 30', the precession of the equinox 50".34 annually, the R. A. of Alcyon (η Tauris) Jan. 1, 1895, 3h. 41m. 15sec. declination 23° 46' 49" N.

24. Proposed by D. H. DAVISON, Minonk, Illinois.

For the purpose of locating the most eligible point for a county seat, it is required to determine the center of a county whose dimensions are as follows: Beginning at the S. W. corner. Thence east 15 miles, thence N. 33$\frac{3}{4}$ miles, thence W. 6 miles to north end of a meridian line running south through the county: thence southwesterly to a point being 6 miles west from the meridian line and 9$\frac{3}{4}$ miles south of its north end. Thence S. 3 miles, thence W. 3 miles, thence S. 21 miles to place of beginning.

[NOTE.—A solution to problem 12 will appear in March number. EDITOR.]

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

DR. HALSTED'S LATEST TRANSLATION.

By LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago, Chicago, Illinois.

It is a matter worthy of remark that, already master of four modern and two dead languages and a translator from several of them of no little repute, Dr. Halsted has vigorously attacked the Russian language with its 36 strange hieroglyphics and now gives to the world an insight into the best scientific thought of Russia, in the shape of a translation of the address by Prof. Vasiliev, President of the Physico-Mathematical Society of Kasan, pronounced last year at the meeting of the Imperial University of Kasan in commemoration of their illustrious compatriot Lobachevsky.

From every one devoted to Mathematics or Philosophy, or indeed to the highest advance of human thought in any form, this address will call forth the deepest admiration for Lobachevsky, now recognized as one of the greatest intellectual revolutionizers the world has ever had. It will arouse a deeper enthusiasm for scientific achievement and widen the horizon of every reader.

Surely no mathematician should miss this gem from fartherest Russia, which, thanks to the rare enthusiasm and energy of Professor Halsted, is easily accessible to all.

A REPLY TO PROFESSOR WHITAKER.

By H. W. DRAUGHON, Olio, Mississippi.

Professor Whitaker, in his reply, devotes a great part of his space to attacking positions which I have never occupied. I will, therefore, consider those points only, in his article, which bear on the subject under discussion. In regard to the expression, $3+\sqrt{2}$, I will state that the sign before $\sqrt{2}$ does not indicate that the positive value of $\sqrt{2}$ is to be taken.

Professor Whitaker does not deny that I find the value of x^2 correctly from the equation, $x^2+2x=3$; he only claims that this equation is not similar to the equation $\sqrt{(x+4)}-\sqrt{(x-4)}=4$. In proof of this, he asserts that, the product of the equation, $x^2+2x-3=0$,and the equations formed by "changing signs" is of the 8th degree. This is not true, when we consider x^2, whose value is required, as the unknown quantity.

I beg leave to remind Professor Whitaker that the performance of this operation on L. B's equation gives an equation of the 2nd degree with reference to $\sqrt{(x+4)}$ or $\sqrt{(x-4)}$. In my example, the produce, is of the 4th degree with reference to x^2. So it appears that the dissimilarity is not great enough to consider, after all.

Let us take a simple equation; $2+x=0\ldots(1)$, for instance. We readily find $x^2=4\ldots(2)$. Now Professor Whitaker claims that the value of x is essentially positive; $\therefore$ from (2) $x=+2$. This value fails to prove when substituted in (1). Let us now multiply as Professor Whitaker suggests, then we have, $(2+x)(2-x)=4-x^2=0$, an equation of the first degree with reference to x^2.

We have here, therefore, an equation which has very much less than "a quarter of a chance of having one root", if we preclude as Professor Whitaker does, negative values of an expression preceded by the sign $+-$, and, at the same time the product of the equation and the equation obtained by "changing signs" is of the first degree.

I will close, by the application of the principle I gave in my former article, to the solution of L. B's equation. We have, $\sqrt{(x+4)}-\sqrt{(x-4}=4\ldots(1)$. Put $x+4=y^2\ldots(2)$, and $x-4=z^2\ldots(3)$; then (1) becomes, $y-z=4\ \ldots(4)$. From (2) and (3), we obtain, $y^2-z^2=8\ldots(5)$. From (4) and (5), we find, $y=3$, and $z=-1$. From (2) or (3) we now find, $x=5$. Now, I do not think there is any hocus-pocus about this. It makes the question clear and enables us to find out everything about the given equation. I feel sure that this view of the matter is in strict accordance with the principles of mathematics.

WANTED.—Dr. G. B. Halsted, Professor J. N. Lyle, Counsellor Dolman, and all other apostles and post-graduate disciples of Lobatschewsky, to inform the numerous readers of the MONTHLY wherein consists *the difference* between the Euclidian Geometry and the Non-Euclidian Geometry. What is *Ideal Space? Hyper-space? Pseudo-spherical*, as used by Professor Lyle in the November MONTHLY.—READER.

NOTE.—Dr. Artemas Martin pointed out to me in a letter (which I misplaced at the time and only recently recovered) that the expressions $m+\sqrt{(2mn)}, n+\sqrt{(2mn)}, m+n+\sqrt{(2mn)}$, given in my article in the January '94 number of the MONTHLY, for the sides of a right triangle, can be reduced to Maseres' expressions, p^2-q^2, $2pq$, p^2+q^2, respectively, by substituting $(p-q)^2$ for m and $2q^2$ for n. I wish to thank Dr. Martin.

LEONARD E. DICKSON, M. A.

EDITORIALS.

THIS number of the MONTHLY was mailed February 28th. It has been cut short, but our readers may look for a good, full number in March.

MARCH Number will be mailed between the 20th and 25th of the month. If you do not get your copy soon after the 25th write to the publishers at once.

IF any of our subscribers have not received any one of the 14 numbers of the MONTHLY already issued write the publishers, and if it is possible, the missing copy will be sent.

AT THE last meeting of the American Mathematical Society, Dr. Macfarlane read a paper on the *Principles of Differentiation in Space-analysis*, which contains among other results, the true generalization for space of Taylor's Theorem. Dr. Macfarlane says, there are many indications pointing to this as the coming subject.

OUR readers will be disappointed because of the absence of the portrait of Professor Chauvenet. No pains were spared on our part in trying to obtain a plate, but our efforts were futile. We may be able to secure a portrait before the end of the year and if we do, will send each of our subscribers a copy, so that it may be bound up with the year's volume.

ERRATA IN PROFESSOR DICKSON'S ARTICLE: p. 9, l. 11 read $s=$ or $<(m-1)/2$; l. 11 read $-A_{m-s}$; p. 40, ls. 34 and 36 for $\pm A_{p-1}$ read $\mp$; l. 35, for $\pm A_{p-2}$ and $\pm A_p$ read $\mp$; l. 37, for $A_1+2=0$ read not$=0$.

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BIOGRAPHY.

PAFNUTIJ LVOVITSCH TCHEBYCHEV.

BY DR. GEORGE BRUCE HALSTED.

OF Russian mathematicians, second only to Lobachevsky should be ranked Pafnutij Lvovitsch Tchebychev. Born in Russia in 1821 and formerly professor at the University of St. Petersburg, he reached deservedly the very highest scientific honors, being privy councillor, the representative of applied mathematics in the Imperial Academy of St. Petersburg, in 1860 made member of the famous Section I.—Geometrie, of the French *Academie des Sciences*, and afterward *Associe Etranger*, the highest honor attainable by a foreigner.

His best known work is the justly celebrated *Memoire Sur les Nombres Premiers*, (Academie imperiale de Saint-Petersbourg, 1850), where he established the existance of limits within which the sum of the logarithms of the primes inferior to a given number must be comprised. This memoir is given in *Liouville's Journal*, 1852, pp. 366-390.

Sylvester afterward contracted Tchebychev's limits; but the original paper remains highly remarkable, especially as it depends on very elementary considerations. In this respect it is in striking contrast to the equally marvelous paper of the lamented Riemann, *Ueber die Anzahl der Primzahlen unter einer gegebenen Groesse* [einer Untersuchung ueber die Haeufigkeit der Primzahlen], presented to the Berlin *Academie* in 1859.

Tchebychev had in 1848 presented a paper with this very title to the St. Petersburg Academie: Sur la totalite des nombres premiers inferieurs a une limite donnee. (Given in Liouville's Journal, 1852, pp. 341-365.) Riemann speaks of the interest long bestowed on this subject by Gauss and Dirichlet, but makes no mention of Tchebychev. However Sylvester speaks of "his usual success in overcoming difficulties insuperable to the rest of the world."

But though best known for his work in the most abstract part of mathematics, in reality Tchebychev was of an eminently practical turn of mind. Thus it was his work "*Theorie des Mecanismes connus sous le nom de parallelogrammes* (Memoires des savants etrangers, Tom. VII.) which led him to the elaborate dissertation *Sur les questions de Minima qui se rattachent a la representation approximative des fonctions*, 91 quarto pages in Memoires de l'Academie Imperiale des sciences de Saint Petersbourg, 1858. While the variable x remains in the vicinity of one same value we can represent with the greatest possible approximation any function $f(x)$, of given form, by the principles of the differential calculus. But this is not the case if the variable x is only required to remain within limits more or less extended. The essentially different methods demanded by this case, which is just the one met in practice, are developed in this memoir. The same line of thought led to his connection with a subject which has since found a place even in elementary text-books, namely rectilineal motion by linkage. He invented a three-bar linkage, which is called Tchebychev's parallel motion, and gives an extraordinarily close approximation to exact rectilineal motion: so much so that in a piece of apparatus exhibited by him in the London Loan Collection of Scientific Apparatus, a plane supported on a combination of two of his parallel motion linkages seemed to have a strictly horizontal movement, though its variation was double that of the tracer in the simple parallel motion.

Tchebychev long occupied himself with attempting to solve the problem of producing exact rectilineal motion by linkage, until he became convinced that it was impossible and even strove long to find a proof of that impossibility. What must have been his astonishment then, when a freshman student of his own class, named Lipkin, showed him the long-sought conversion of circular into straight motion.

Tchebychev brought Lipkin's name before the Russian government, and secured for him a substantial reward for his supposed original discovery.

And perhaps it was independent, but it had been found several years previously by a French lieutenant of engineers, Peaucellier, and first published by him in the form of a question in the '*Annales de Mathematique* in 1864. When Tchebychev was on a visit to London, Sylvester inquired after the progress of his proof of the impossibility of exact parallel motion, when the Russian announced its double discovery and made a drawing of the cell and mounting. This Sylvester happened to show to Manuel Garcia, inventor of the laryngoscope, and the next day received from him a model constructed of pieces of wood fastened with nails as pivots, which, rough as it was, worked perfectly. Sylvester exhibited this to the Philosophical Club of the Royal Society and in the Athenaeum Club, where it delighted Sir Wm. Thomson, now Lord Kelvin, and led to the extraordinary lecture *On recent Discoveries in Mechanical Conversion of Motion*, delivered by Sylvester before the Royal Institution on January 23, 1874. This in turn led to Kempe's remarkable developement of the subject, and to Hart's discovery of a five-bar linkage which does the same work as Peaucellier's of seven. Henceforth Peaucellier's Cell and Hart's Contraparallelogram will take their place in our text-books of geometry, and straight lines can be drawn without begging the question by assuming first a straight edge or ruler as does Euclid. Thus Kempe's charming book, *How to draw a straight line*, is a direct outcome of Tchebychev's sketch for Sylvester, to whom, in parting, he used the characteristic words: "Take to kinematics, it will repay you; it is more fecund than geometry; it adds a fourth dimension to space." As might perhaps have been expected, the immortal Lobachevsky found in his compatriot a devoted admirer. Not only was Tchebychev an active member of the Committee of the Lobachevsky-fund, but he took the deepest interest in all connected with the spread of the profound ideas typified in the Non-Euclidean geometry.

Knowing this, Vasiliev in his last letter asked that a copy of my translation of his Address on Lobachevsky be forwarded to the great man. His active participation in scientific assemblies is also worthy of note: for example at the 'Congres de l'Association francaise pour l'avancement des sciences, a Lyon' he read two interesting papers, *Sur les valeurs limites des integrales*, and *Sur les quadratures*, both afterwards published in *Liouville's Journal*.

LAGRANGE'S GENERALIZED EQUATIONS OF MOTION.

By F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Lagrange's celebrated equations of motion, as given in his *Mecanique Analytique*, consist of *a transformation* from Cartesian to generalized co-ordinates, *of the indeterminate equation of motion*. In order to avoid the unnecessary complication incident to the introduction of such indeterminate variations, as δx, δy, δz, etc., in the derivation of these equations of motion, we must have recourse to the Cartesian equations of the unconstrained motion of particles.

Assume $(x_1,y_1,z_1),\ldots,(x_n,y_n,z_n)$ as the rectilineal-rectangular co-ordinates of the material particles $m_1,\ldots,m_n$; also, assume $(X_1, Y_1, Z_1),\ldots,(X_n, Y_n, Z_n)$ as the force components of the particles. Make $(\psi_1,\phi_1,\theta_1),\ldots,(\psi_n,\phi_n,\theta_n)$ the generalized co-ordinates of the *respective* particles at any instant of time; that is, let ψ_1,ϕ_1,θ_1, be regarded as determinate functions of x_1,y_1,z_1, respectively,—or vice versa. This reciprocal determinateness is to be a characteristic of all the material particles. The velocity-components in

Cartesian co-ordinates are $(dx_1 / dt, dy_1 / dt, dz_1 / dt)$, etc.; while the corresponding generalized velocity-components are $(d\psi_1 / dt, d\phi_1 / dt, d\theta_1 / dt)$, etc. Similarly the acceleration-components in Cartesian co-ordinates are $(d^2x_1 / dt^2, d^2y_1 / dt^2, d^2z_1 / dt^2)$, etc.; while the corresponding generalized acceleration-components are $(d^2\psi_1 / dt^2, d^2\phi_1 / dt^2, d^2\theta_1 / dt^2)$, etc.

From well-known principles of the differential calculus,

$$\frac{dx_1}{dt} = \frac{dx_1}{d\psi_1}\cdot\frac{d\psi_1}{dt} + \frac{dx_1}{d\phi_1}\cdot\frac{d\phi_1}{dt} + \frac{dx_1}{d\theta_1}\cdot\frac{d\theta_1}{dt} \ldots (A_1),$$

$$\frac{dy_1}{dt} = \frac{dy_1}{d\psi_1}\cdot\frac{d\psi_1}{dt} + \frac{dy_1}{d\phi_1}\cdot\frac{d\phi_1}{dt} + \frac{dy_1}{d\theta_1}\cdot\frac{d\theta_1}{dt} \ldots (A_2),$$

$$\frac{dz_1}{dt} = \frac{dz_1}{d\psi_1}\cdot\frac{d\psi_1}{dt} + \frac{dz_1}{d\phi_1}\cdot\frac{d\phi_1}{dt} + \frac{dz_1}{d\theta_1}\cdot\frac{d\theta_1}{dt} \ldots (A_3).$$

Since the kinetic energy of the particle m_1 represented by the Cartesian co-ordinates (x_1, y_1, z_1) becomes

$$K_1 = \tfrac{1}{2}m_1[(dx_1 / dt)^2 + (dy_1 / dt)^2 + (dz_1 / dt)^2] \ldots (a),$$

it is obvious [substituting the square of (A_1), of (A_2), and of (A_3), in (a)] that the corresponding kinetic energy expressed in terms of generalized co-ordinates may be represented (according to the usual system of functional notation) by

$$T_1 = \tfrac{1}{2}\left[(\psi_1,\psi_1)\left(\frac{d\psi_1}{dt}\right)^2 + (\phi_1,\phi_1)\left(\frac{d\phi_1}{dt}\right)^2 + (\theta_1,\theta_1)\left(\frac{d\theta_1}{dt}\right)^2\right.$$

$$\left. + 2(\psi_1,\phi_1)\left(\frac{d\psi_1}{dt}\cdot\frac{d\phi_1}{dt}\right) + 2(\phi_1,\theta_1)\left(\frac{d\phi_1}{dt}\cdot\frac{d\theta_1}{dt}\right) + 2(\theta_1,\psi_1)\left(\frac{d\theta_1}{dt}\cdot\frac{d\psi_1}{dt}\right)\right] \ldots (a').$$

The *coefficients* in (a'), *inclosed by the smaller parenthesis*, are homogeneous functions of the co-odinates. These coefficients are determinable from system-conditions. With respect to these coefficients, the necessary condition is that they must give a finite and positive value of T_1, for whatever values may be assigned to the variables.

If $(\delta x_1, \delta y_1, \delta z_1), \ldots, (\delta x_n, \delta y_n, \delta z_n)$ represent the components of any infinitely small motions possible without breaking the conditions of the material system of which $m_1, \ldots, m_n$ are the component particles: then *the work done by the forces* whose components have already been specified, *upon the particle* m_1, becomes $W_1 = X_1\delta x_1 + Y_1\delta y_1 + Z_1\delta z_1 \ldots (b)$.

In order to deduce the expression for W_1 in terms of generalized co-ordinates, we have from the differential calculus:

$$\delta x_1 = \frac{dx_1}{d\psi_1}\delta\psi_1 + \frac{dx_1}{d\phi_1}\delta\phi_1 + \frac{dx_1}{d\theta_1}\delta\theta_1 \ldots (B_1),$$

$$\delta y_1 = \frac{dy_1}{d\psi_1}\delta\psi_1 + \frac{dy_1}{d\phi_1}\delta\phi_1 + \frac{dy_1}{d\theta_1}\delta\theta_1 \ldots (B_2),$$

$$\delta z_1 = \frac{dz_1}{d\psi_1} + \frac{dz_1}{d\phi_1}\delta\phi_1 + \frac{dz_1}{d\theta_1}\delta\theta_1 \ldots (B_3).$$

Multiplying (B_1) by X_1, (B_2) by Y_1, (B_3) by Z_1, and adding products, we have

$$X_1\delta x_1 + Y_1\delta y_1 + Z_1\delta z_1 = \left(X_1\frac{dx_1}{d\psi_1} + Y_1\frac{dy_1}{d\psi_1} + Z_1\frac{dz_1}{d\psi_1}\right)\delta\psi_1$$

$$\left(X_1\frac{dx_1}{d\phi_1} + Y_1\frac{dy_1}{d\phi_1} + Z_1\frac{dz_1}{d\phi_1}\right)\delta\phi_1 + \left(X_1\frac{dx_1}{d\theta_1} + Y_1\frac{dy_1}{d\theta_1} + Z_1\frac{dz_1}{d\theta_1}\right)\delta\theta_1,$$

$= \Psi_1\delta\psi_1 + \Phi_1\delta\phi_1 + \Theta_1\delta\theta_1 \ldots (d_1)$, in which the coefficients of the indeterminate variations are the generalized force-components with respect to the particle m_1. According to D'Alembert's principle, the Cartesian *Equations of motion* in the order of the particles specified, become

$$X_1 = m_1\frac{d^2x_1}{dt^2},\ Y_1 = m_1\frac{d^2y_1}{dt^2},\ Z_1 = m_1\frac{d^2z_1}{dt^2} \ldots\ldots$$

$$X_n = m_n\frac{d^2x_n}{dt^2},\ Y_n = m_n\frac{d^2y_n}{dt^2},\ Z_n = m_n\frac{d^2z_n}{dt^2}.$$

Multiplying these equations, respectively, by

$\frac{dx_1}{d\psi_1}, \frac{dy_1}{d\psi_1}, \frac{dz_1}{d\psi_1} \ldots\ldots \frac{dx_n}{d\psi_n}, \frac{dy_n}{d\psi_n}, \frac{dz_n}{d\psi_n}$; and then adding the products, we have the equations,

$$\Psi = m_1\left(\frac{d^2x_1}{dt^2}\cdot\frac{dx_1}{d\psi_1} + \frac{d^2y_1}{dt^2}\cdot\frac{dy_1}{d\psi_1} + \frac{d^2z_1}{dt^2}\cdot\frac{dz_1}{d\psi_1}\right) + \ldots$$

$$+ m_n\left(\frac{d^2x_n}{dt^2}\cdot\frac{dx_n}{d\psi_n} + \frac{d^2y_n}{dt^2}\cdot\frac{dy_n}{d\psi_n} + \frac{d^2z_1}{dt^2}\cdot\frac{dz_1}{d\psi_n}\right)\ldots (c_1).$$

After performing similar operations, we have the equations:

$$\Phi = m_1\left(\frac{d^2x_1}{dt^2}\cdot\frac{dx_1}{d\phi_1} + \frac{d^2y_1}{dt^2}\cdot\frac{dy_1}{d\phi_1} + \frac{d^2z_1}{dt^2}\cdot\frac{dz_1}{d\phi_1}\right) + \ldots$$

$$+ m_n\left(\frac{d^2x_n}{dt^2}\cdot\frac{dx_n}{d\phi_n} + \frac{d^2y_n}{dt^2}\cdot\frac{dy_n}{d\phi_n} + \frac{d^2z_n}{dt^2}\cdot\frac{dz_n}{d\phi_n}\right)\ldots (c_2),$$

and $$\Theta = m_1\left(\frac{d^2x_1}{dt^2}\cdot\frac{dx_1}{d\theta_1} + \frac{d^2y_1}{dt^2}\cdot\frac{dy_1}{d\theta_1} + \frac{d^2z_1}{dt^2}\cdot\frac{dz_1}{d\theta_1}\right) + \ldots$$

$$+ m_n\left(\frac{d^2x_n}{dt^2}\cdot\frac{dx_n}{d\theta_n} + \frac{d^2y_n}{dt^2}\cdot\frac{dy_n}{d\theta_n} + \frac{d^2z_n}{dt^2}\cdot\frac{dz_n}{d\theta_n}\right)\ldots (c_3).$$

Taken in its utmost generality; that is, in case of *isolated* and *con-*

servative Material-systems, (*a*) becomes $K=\Sigma\frac{1}{2}m[(dx/dt)^2+(dy/dt)^2+(dz/dt)^2]\ldots(a_1)$: also, from ($a'$), in case of analogous systems,

$$T=\frac{1}{2}\left[(\psi,\psi)\left(\frac{d\psi}{dt}\right)^2+(\phi,\phi)\left(\frac{d\phi}{dt}\right)^2+(\theta,\theta)\left(\frac{d\theta}{dt}\right)^2\right.$$

$$\left.+2(\psi,\phi)\left(\frac{d\psi}{dt}\cdot\frac{d\phi}{dt}\right)+2(\phi,\theta)\left(\frac{d\phi}{dt}\cdot\frac{d\theta}{dt}\right)+2(\theta,\psi)\left(\frac{d\theta}{dt}\cdot\frac{d\psi}{dt}\right)\right]\ldots(a'_1).$$

Supposing dx_1/dt to be a function of the Cartesian co-ordinates, and also a function of the generalized velocity-components, we have from the differential calculus:

$$\frac{d^2x_1}{dt^2}\cdot\frac{dx_1}{d\psi_1}=\frac{d}{dt}\left(\frac{dx_1}{dt}\cdot\frac{dx_1}{d\psi_1}\right)-\frac{dx_1}{dt}\cdot\frac{d}{dt}\left(\frac{dx_1}{d\psi_1}\right),$$

$$=\frac{d}{dt}\left[\frac{dx_1}{dt}\left(\frac{dx_1}{dt}\Big/\frac{d\psi_1}{dt}\right)\right]-\frac{dx_1}{dt}\left(\frac{dx_1}{dt}\Big/d\psi_1\right).$$

$$=\frac{d}{dt}\left[\frac{1}{2}\text{ of }\frac{d[(dx_1/dt)^2]}{d(d\psi_1/dt)}\right]-\frac{1}{2}\text{ of }\frac{d[(dx_1/dt)^2]}{d\psi_1}\ldots(c).$$

For *every* term of (c_1), (c_2), and (c_3) *the proper expression* can be written by analogy, from the right-hand member of (c).

Transforming (c_1), (c_2), and (c_3), by means of (c) *generally applied* and then using T for the kinetic energy of the system, we have

$$\frac{d}{dt}\left(dT\Big/\frac{d\psi}{dt}\right)-\frac{dT}{d\psi}=\Psi,\ \frac{d}{dt}\left(dT\Big/\frac{d\phi}{dt}\right)-\frac{dT}{d\phi}=\Phi,$$

and $\frac{d}{dt}\left(dT\Big/\frac{d\theta}{dt}\right)-\frac{dT}{d\theta}=\Theta,$

which are Lagrange's equations of motion in terms of generalized co-ordinates.

With respect to any isolated and conservative material-system, the generalization of (b_1) gives $\Sigma(X\delta x+Y\delta y+Z\delta z)=\Psi\delta\psi+\Phi\delta\phi+\Theta\delta\theta\ldots(d)$.

The potential energy of a conservative system is a function of the co-ordinates by which the different positions of the various parts of such a system are specified. With reference to the configuration which an isolated and conservative material-system has at any instant,the potential energy represents the amount of work required to bring the system to that configuration against its mutual forces during the passage of the system from any one chosen configuration to the configuration referred to at the time. Hence if an aggregation of moving particles constitutes an isolated and conservative material system whose potential energy in the configuration specified by the Cartesian co-ordinates x, y, z) is represented by V, we must have $\delta V=-\Sigma(X\delta x+Y\delta y+Z\delta z)\ldots(e)$.

From (d) and (e), $\Psi\delta\psi+\Phi\delta\phi+\Theta\delta\theta=-\delta V\ldots(f)$.

$$\therefore\quad \Psi=-\frac{dV}{d\psi},\quad \Phi=-\frac{dV}{d\phi},\ \text{and}\ \Theta=-\frac{dV}{d\theta}.$$

Therefore the Lagrangian equations of motion may be written:

$$\frac{d}{dt}\left(dT\Big/\frac{d\psi}{dt}\right)=\frac{dT}{d\psi}-\frac{dV}{d\psi},\quad \frac{d}{dt}\left(dT\Big/\frac{d\phi}{dt}\right)=\frac{dT}{d\phi}-\frac{dV}{d\phi},$$

$$\text{and}\ \frac{d}{dt}\left(dT\Big/\frac{d\theta}{dt}\right)=\frac{dT}{d\theta}-\frac{dV}{d\theta}.$$

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

BY GEORGE BRUCE HALSTED, A. M., (Princeton) Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the February Number].

PROPOSITION XIV. *The hypothesis of obtuse angle is inconsistent with Euclid's assumption: Two straight lines cannot enclose a space.*

Proof. From the hypthesis of obtuse angle, assumed as true, [and the first 28 propositions of Euclid], we have now deduced the truth of Euclid's Postulatum; that two straights will meet each other in some point toward those parts, toward which a certain straight, cutting them, makes two internal angles, of whatever kind, less than two right angles.

But this Postulatum holding good, on which Euclid supports himself after the twenty-eighth proposition of his first book, it is manifest to all Geometers, that the hypothesis of right angle alone is true, nor any place left for the hypothesis of obtuse angle. Therefore the hypothesis of obtuse angle is inconsistent with Euclid's assumption. Quod erat demonstrandum.

Otherwise, and more immediately.

Since from the hypothesis of obtuse angle we have demonstrated (P. IX.) that two (fig. 11.) acute angles of the triangle APX, right-angled at P, are greater than one right angle; it follows that an acute angle PAD may be assumed such, that together with the aforesaid two acute angles it makes up two right angles. But then the straight AD must (by the preceding proposition,

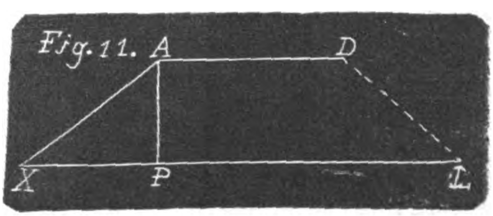
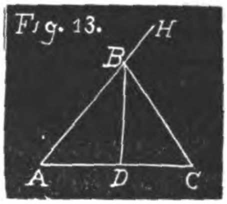


Fig. 11.

joined to the hypothesis of obtuse angle) at length meet with this PL, or XL, regard being had to the secant, or incident AP; which is manifestly absurd against Eu. I. 17, if we regard the secant or incident AX.

PROPOSITION XV. *By any triangle ABC, of which the three angles (fig. 13.) are together equal to, or greater, or less than two right angles, is established respectively the hypothesis of right angle, or obtuse angle, or acute angle.*

Proof. For anyhow two angles of this triangle, as suppose A, and C, will be acute, because of Eu. I. 17. Wherefore the perpendicular, let fall from the apex of the remaining angle B upon this AC, will cut this AC (Eu. I. 17.) in some intermediate point D.



If therefore three angles of this triangle ABC are supposed equal to two right angles, it follows that all the abgles of the triangles ADB, CDB will be together equal to four right angles, because of the two additional right angles at the point D. This holding good, now of neither of the said triangles, as suppose ADB, will the three angles together be less, or greater than two right angles; for thus viceversa the three angles together of the other triangle would be greater, or less than two right angles. Wherefore (P. IX.) from one triangle indeed would be established the hypothesis of acute angle, and from the other the hypothesis of obtuse angle; which is contrary to P. VI. and P. VII.

Therefore the three angles together of either of the aforesaid triangles will be equal to two right angles; and therefore (P. IX.) is established the hypothesis of right angle. Quod erat primo loco demonstrandum.

But if however the three angles of the proposed triangle ABC are taken greater than two right angles; now of the two triangles ADB, CDB all the angles together will be greater than four rignt angles, because of the two additional right angles at the point D.

This holding good; now of neither of the said triangles will the three angles together be precisely equal to, or less than two right angles; for thus viceversa the three angles of the other triangle would be together greater than two right angles. Wherefore (P.IX.) from one triangle indeed would be established the hypothesis either of right angle or of acute angle, and from the other the hypothesis of obtuse angle, which is contrary to P.V, P.VI, and P.VII.

Therefore the three angles together of either of the aforesaid triangles will be greater than two right angles; and therefore (P.IX.) is established the hypothesis of obtuse angle. Quod erat secundo loco demonstrandum.

But finally. If the three angles of the proposed triangle ABC are taken less than two right angles, now of the two triangles ADB, CDB, all the angles together will be less than four right angles, because of the two additional right angles at the point D.

This holding good; now of neither of the said triangles will the three angles together be equal to, or greater than two right angles; for thus viceversa of the other triangle the three angles together would be less than two right angles. Wherefore (P.IX.) from one triangle indeed would be established the hypothesis either of right angle or obtuse angle, and from the other the hypothesis of acute angle; which is contrary to P.V, P.VI, and P.VII.

Therefore the three angles together of either of the aforesaid triangles will be less than two right angles; and therefore (P.IX.) is established the hypothesis of acute angle. Quod erat tertio loco demonstrandum.

Accordingly by any triangle ABC, of which the three angles are equal to, or greater, or less than two right angles, is established respectively the hypothesis of right angle, or obtuse angle, or acute angle. Quod erat propositum.

COROLLARY. Hence, any one side of any proposed triangle being produced, as suppose AB to H, the external angle HBC will be (Eu.I.13.) either equal to, or less, or greater than the remaining internal and opposite angles together at the points A and C, according as is true the hypothesis of right angle, or obtuse angle, or acute angle. And inversely.

[To be continued.]

THE "IRREDUCIBLE CASE."

By J. K. ELLWOOD, A. M., Principal Colfax School, Pittsburg, Pa.

PROBLEM.—*To extract the cube root of* $a \pm \sqrt{-b}$.

Put $\sqrt[3]{a+\sqrt{-b}}=m+n$, and $\sqrt[3]{a-\sqrt{-b}}=m-n$.

Then $a+\sqrt{-b}=m^3+3m^2n+3mn^2+n^3$, and $a-\sqrt{-b}=m^3-3m^2n+3mn^2-n^3$. Hence $a=m^3+3mn^2$, and $\sqrt{-b}=3m^2n+n^3$.

Example 1. Find the cube root of $9+25\sqrt{-2}$.

Here $a=m^3+3mn^2=9=3^3-18$. Hence $3mn^2=-18$, and $n=\sqrt{-2}$.

To verify these values of m and n substitute them in $\sqrt{-b}=3m^2n+n^3=25\sqrt{-2}$. Doing this we have $27\sqrt{-2}+(-2\sqrt{-2})=25\sqrt{-2}$.

$\therefore$ $3+\sqrt{-2}$ is the required root. When the substituted values of m and n do not give the second term they are not correct, and other values must be found by trial.

Example 2. Find the cube root of $2\sqrt{11}+30\sqrt{-3}$. Here $a=m^3+3mn^2=2\sqrt{11}=(\sqrt{11})^3-9\sqrt{11}$.

Hence $3mn^2=-9\sqrt{11}$, and $n=\sqrt{-3}$. Since these values of m and n substituted in $3m^2n+n^3$ give $\sqrt{-b}=30\sqrt{-3}$, the root is $\sqrt{11}+\sqrt{-3}$.

This method frequently enables us to simplify Cardan's formula for cubics in what is called the "irreducible case", said formula being

$$x=\sqrt[3]{q+\sqrt{(q^2-p^3)}}+\sqrt[3]{q-\sqrt{(q^2-p^3)}}.$$

1. * $x^3-22x-24=0$. Here $p=\frac{22}{3}$, $q=12$.

Then $x=\sqrt[3]{12+\frac{26}{3}\sqrt{(-\frac{10}{3})}}+\sqrt[3]{12-\frac{26}{3}\sqrt{(-\frac{10}{3})}}$.

Now, $a=m^3+3mn^2=12=-2^3+20$. $3mn^2=20$, $n=\sqrt{-\frac{10}{3}}$.

$\therefore$ $-2\pm\sqrt{-\frac{10}{3}}$ are the roots. Then $x=(-2+\sqrt{-\frac{10}{3}})+(-2-\sqrt{-\frac{10}{3}})=-4$.

2. $x^3-8x^2+19x-12=0$. Put $x=y+\frac{8}{3}$.

Then $y^3-\frac{7}{3}y+\frac{20}{27}=0$, where $p=\frac{7}{9}$, $q=-\frac{10}{27}$.

Then $y=\sqrt[3]{-\frac{10}{27}+\sqrt{(-\frac{243}{729})}}+\sqrt[3]{-\frac{10}{27}-\sqrt{(-\frac{243}{729})}}$

$=\sqrt[3]{-\frac{10}{27}+\frac{1}{3}\sqrt{-3}}+\sqrt[3]{-\frac{10}{27}-\frac{1}{3}\sqrt{-3}}$

Here $a=m^3+3mn^2=-\frac{10}{27}=(\frac{2}{3})^3-\frac{18}{27}$. Hence $n=\sqrt{-\frac{1}{3}}=\frac{1}{3}\sqrt{-3}$.

$\therefore$ $y=(\frac{2}{3}+\frac{1}{3}\sqrt{-3})+(\frac{2}{3}-\frac{1}{3}\sqrt{-3})=\frac{4}{3}$. Then $x=\frac{4}{3}+\frac{8}{3}=4$.

3. $x^3-12x^2+41x-42=0$. Put $x=y+4$.

Then $y^3-7y=6$, where $p=\frac{7}{3}$, $q=3$. Then $y=\sqrt[3]{3+\frac{10}{9}\sqrt{-3}}+\sqrt[3]{3-\frac{10}{9}\sqrt{-3}}$.

Here $a=3=(\frac{3}{2})^3-\frac{3}{8}$. $\therefore$ $n=\frac{1}{6}\sqrt{-3}$.

$\therefore$ $y=(\frac{3}{2}+\frac{1}{6}\sqrt{-3})+(\frac{3}{2}-\frac{1}{6}\sqrt{-3})=3$, and $x=3+4=7$.

* Maynard in his Key to Bonnycastle's Introduction, page 78, says this "can only be resolved by a table of sines, or by infinite series."

THEOREM 16 OF LOBATSCHEWSKY'S THEORY OF PARALLELS.

By JOHN N. LYLE, Ph. D., Professor of Natural Sciences, Westminster College, Fulton, Missouri

Says Lobatschewsky in his Theorem 16—"All straight lines which in a plane go out from a point can, with reference to a given straight line in the same plane, be divided into two classes—into *cutting* and *not-cutting*. The *boundary lines* of the one and the other class of those lines will be called *parallel to the given line*.

From the point A (Fig. 1) let fall upon the line BC the perpendicular AD, to which again draw the perpendicular AE. In the right angle EAD either will all all straight lines which go out from the point A meet the line DC, as for example AF, or some of them, like the perpendicular AE, will not meet the line DC. In the uncertainty whether the perpendicular AE is the only line which does not meet DC, we will assume it may be possible that there are still other lines, for

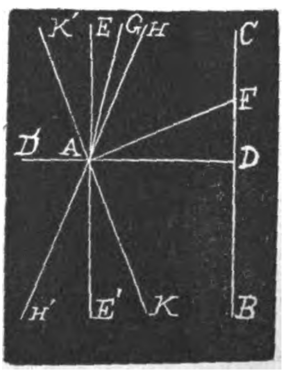


Fig. 1.

example AG, which do not cut DC, how far soever they may be prolonged. In passing over from the cutting lines, as AF, to the not-cutting lines, as AG, we must come upon a line AH, parallel to DC, a boundary line, upon one side of which all lines AG are such as do not meet the line DC, while upon the other side every straight line AF cuts the line DC.

The angle HAD between the parallel HA and the perpendicular AD is called the parallel angle (angle of parallelism), which we will here designate by $\Pi(p)$ for $AD=p$."

Does Lobatschewsky class his boundary line AH among the *cutting* or the *not-cutting* lines? Evidently among the cutting lines, for under Theorem 33, referring to his equation $S'=se^{-x}$, he says "We may here remark, that $S'=0$ for $x=\infty$, hence not only does the distance between two parallels decrease (Theorem 24), but with the pralongation of the parallels towards the side of the parallelism this at last wholly vanishes."

Agreeably to this assumption of Lobatschewsky let y be the point in space at which the decreasing distance between his parallel lines AH and DC wholly vanishes. According to Euclid's postulate 2 the terminated line Dy may be extended beyond the point y. If this is not permitted, Euclid's postulate 2 would be discredited in Lobatschewsky's geometry. Assume that Euclid's postulates hold everywhere in space. On the basis of that assumption we have the authority to locate any point as z beyond y on Dy extended. Then the point z is within the angle yAE. From z draw a straight line to the point A. This must be permitted, otherwise postulate 1 would be discredited in Lobatschewsky's geometry. Since z is within the angle yAE the straight line Az must fall between Ay and AE.

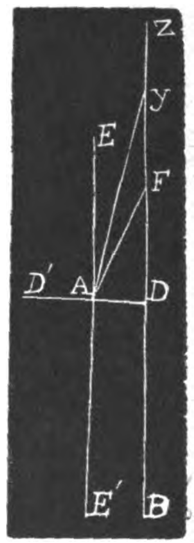


Fig. 2.

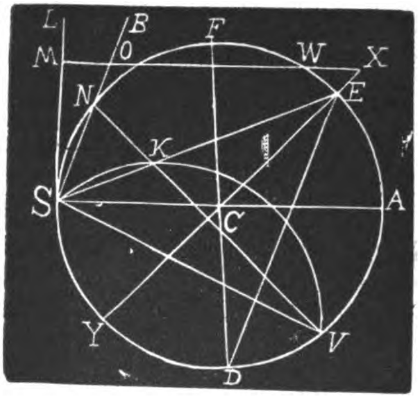
But since by Lobatschewsky's hypothesis no straight line between AHy and AE can meet DC produced, the line Az must fall between Ay and AD. That is, Az must lie on both sides of Ay at the same time. Says W. K. Clifford, an enthusiastic admirer of Lobatschewsky's Imaginary Geometry—"but the way things come out of one onother is quite lovely."

SOME FALLACIES OF AN ANGLE TRISECTOR.

By LEONARD E. DICKSON, M. A., Fellow in Mathematics, The University of Chicago.

Since 1860, Mr. L. S. Benson of New York City has labored to throw discredit upon vital parts of mathematics. He easily trisects any angle by simple geometry and because the results following from his construction are not in harmony with results derived from Trigonometry, he quietly casts overboard as false all Trigonometry and with it the applied mathematics!

His trisection, however, really offers an interesting fallacy especially from a graphical standpoint. The *gist* of the construction given in his circulars "Mathematics out of joint," etc., is as follows:



To trisect any angle BSA between 60° and 90°, construct the right angle LSA and a circle with any point C on SA as center. Make arc $AW=60^\circ$ and draw $WM \parallel SA$. Take $WX=MO$ and draw diameter $XECY$. Draw chord $ED \parallel SB$ and diameter DF. Then arc NA is trisected by F and E. Proof: arcs NE, SD, FA are equal, also $FE=YD$, also $NF=SY=EA$. Draw chord $DV \parallel SE$. Describe circle SKV with radius $=SC$ through points S and V. Then arc $SN=$ arc SK, each being double the measure of $\angle NVS$; similarly, $\angle ESV$ makes arc $KV=$ arc EV; $\angle NVE$ makes arc $KV=$ arc NE. Since arcs $SKV=SDV$ and $KV=EV=SD$, then $SN=SK=DV$. $\therefore$ arc $SNE=$ arc EVD and finally arc $SY=$ arc YD. $\therefore$ arcs $NF=FE=EA$.

Since 3 arc $AW=180^\circ=3$ (arc $AE+$ arc EW) $=$ arc $AN+3$ arc EW, we find arc $SN=3$ arc EW. Thus any angle $<90^\circ$ is trisected.

The step used by Mr. Benson which lacks proof is the fact that NV and SE intersect on the arc drawn through S and V with radius $=SC$. It can be shown by Trigonometry that this fact is true only for one particular angle $67\frac{1}{2}^\circ$. But the error is so small that it is scarcely apparent in an ordinary-sized figure, even if drawn accurately. But with a larger figure and the favorable case of an angle near 60° or 90°, the error is distinct. The deception is increased by the fact that to a certain difference in the lengths of the arcs NF and FE corresponds a much smaller distance between K_1 (the intersection of NV and arc SV) and K_2 (the intersection of SE and arc SV). In "another proof," Mr. Benson attempts to prove that K_1 and K_2 coincide (in the point K of the figure) but lands immediately in a rediculous *argumento in circulo.*

ARITHMETIC.

Conducted by B.F.FINKEL, Kidder, Missouri. All Contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

35. **Proposed by B. F. FINKEL, Professor of Mathematics in Kidder Institute, Kidder, Missouri.**

Between Sing-Sing and Tarry-Town, I met my worthy friend, John Brown.
And seven daughters, riding nags, and every one had seven bags:
In every bag were thirty cats, and every cat had forty rats.
Besides a brood of fifty kittens. All *but* the nags were wearing mittens!
Mittens, kittens—cats, rats—bags, nags—Browns,
How many were met between the towns?

[From *Mattoon's Common Arithmetic*].

II. **Solution by T. W. PALMER, Professor of Mathematics, University, Alabama.**

1. $8=$No. of Browns met.
2. $8=8\times1=$No. of nags.
3. $112=16\times7=$No. of bags, (each bag and each Brown had 7).
4. $3360=112\times30=$No. of cats.
5. $134400=3360\times40=$No. of rats.
6. $168000=3360\times50=$No. of kittens.
7. $16=8\times2=$No. of mittens worn by Browns.
8. $13440=3360\times4=$No. of mittens worn by cats.
9. $537600=134400\times4=$No. of mittens worn by rats.
10. $672000=168000\times4=$No. of mittens worn by kittens.

$1528944=$Browns+nags+bags+cats+rats+kittens+mittens.

NOTE—Mr. Horn in January Number has probably given correct solution, but the language of the example will admit of the above interpretation. *T. W. P.*

Remark on Solution of Number 35 by COOPER D. SCHMITT, Knoxville, Tennessee.

Mr. Horn's addition is not correct to obtain line numbered 7. I would suggest that as cats, rats and kittens have *four* legs each that *four* mittens be assigned to each, this will make the answer to the problem. 764488. I can not see Mr. Mattoon's interpretation of the problem.

38. **Proposed by J. A. CALDERHEAD, B. Sc., Superintendent of Schools, Limaville, Ohio.**

What must be the thickness of a 36-inch shell, in order that it may weigh 1 ton: supposing a 13-inch shell to weigh 200 pounds, when two inches thick?

III. **Solution by P. S. BERG, Apple Creek, Ohio.**

$\frac{13^3\pi}{6}-\frac{9^3\pi}{6}=\frac{1468\pi}{6}$, solid contents of 13-inch shell.

$\frac{1468\pi}{6}\times10=\frac{14680\pi}{6}$, solid contents of 36-inch shell

$\frac{36^3\pi}{6}-\frac{14680\pi}{6}=\frac{31976\pi}{6}$, volume of hollow within 36-inch shell

$\sqrt[3]{\frac{31976\pi}{6}\div\frac{\pi}{6}}=31.736$, diameter of hollow within 36-inch shell.

$(36-31.736)\div 2=2.132$ in. thickness of shell.

This problem was solved with same result, by *Hon. Josiah H. Drummond, J. F. W. Scheffer, Frank Horn J. K. Ellwood, and Cooper D. Schmitt.*

39. Proposed by P. C. CULLEN, Superintendent of Schools, Brady, Nebraska.

A, B, and C start from same point at same time. A north at rate of three miles per hour, B east at rate of four miles and C west at rate of five miles per hour. B at end of two hours starts at such an angle as to intersect A. How long after starting must C start north-west in order to meet A and B at common point?

Solution by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Maine, and J. W. WATSON, Middle Creek, Ohio.

Let x be the time after B turns till he meets A. The route of both is a right angle triangle with base 8; perpendicular $3x+6$, and hypotenuse $4x$. Hence, $16x^2=(3x+6)^2+64$, whence $x=7\frac{1}{7}$ or -2. But the -2 value makes them turn back and meet at point of starting. Let y=time before C turns. Then $7\frac{1}{7}+2-y$=time after he turns. $3x+6=\frac{192}{7}$=perpendicular, $5y$=base, and $5(\frac{64}{7}-y)$=hypotenuse. Hence, $25y^2+(\frac{192}{7})^2=25(\frac{64}{7}-y)^2$, whence $y=2\frac{162}{175}$ hours.

Excellent solutions of this problem were received from *G. B. M. Zerr, P. S. Berg, J. K. Ellwood, Cooper D. Schmitt, and J. F. W. Scheffer.*

40. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Find the market-price of $m=3\%$-stock, in order that it may yield $n=3\frac{1}{3}\%$ interest after deducting $d=\$\frac{7}{20}$ from every $S=\$12$.

Solution by the PROPOSER.

According to the conditions of the problem, the deduction from the the par (\$100) value of a share is $100d/S$ dollars, $=\$2\frac{11}{12}$; therefore, $100(1-d/S)$ dollars are to yield $\$m$ interest. In order to yield $\$n$ interest,

the market-price must be $P=100\left(\frac{m}{n}\right)\left(1-\frac{d}{S}\right)$dollars, $=\$87\frac{3}{8}$.

COR.—Put $m=n$; then $P=\$97\frac{1}{12}$, which is the correct result of this problem as proposed in the December, '94, MONTHLY.—*F. P. M.*

41. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

If I gain \$2 in \$5 by selling a horse for \$150, what per cent. would I gain by selling the horse for \$120?

Solution by P. S. BERG, Apple Creek, Ohio, and the PROPOSER.

Since gaining \$2 in \$5 is gaining 40%, the cost of the horse is $\$107\frac{1}{7}$. Hence the gain required is 12%.

PROBLEMS.

46. Proposed by T. W. PALMER, Professor of Mathematics, University of Alabama.

A borrows \$500.00 from a Building and Loan Association and agrees to pay

\$9.50 per month for 72 months, the first payment to be made at the end of the first month. What rate of interest does *A* pay? The Association claims to charge only 8 per cent. (the legal rate in Alabama). How can 8 per cent. be figured out on the above.

47. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Mr. Merchant sells 20% above cost, with weights and measures $12\frac{1}{2}$% "short," allows a discount of \$5 on every bill of \$50, and loses 5% of his sales as "bad debts." Find his *rate per cent* of net profit, or net loss; one cent in every dollar of sale proves counterfit, and collection-charges are $2\frac{1}{2}$%.

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

36. Proposed by J. A. CALDERHEAD, B. Sc., Superintendent of Schools, Limaville, Ohio.

Resolve $(x^2+y^1)(x^2+z^2)(y^2+z^2)$ into the sum of two squares.

I. Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

By Euler's theorem we have

$$(x^2+y^2)(x^2+z^2)=(x^2+yz)^2+(xz\mp xy)^2=A^2+B^2.$$

$$(x^2+y^1)(x^2+z^2)(y^2+z^2)=(A^2+B^2)(y^2+z^2)=(Ay\pm Bz)^2+(Az\mp By)^2$$

$$=\{(x^2\pm yz)y\pm(z\mp y)xz\}^2+\{(x^2\pm yz)z\mp(z\mp y)xy\}^2.$$

$\therefore$ the sum of two squares in four ways.

II. Solution by the PROPOSER.

By determinants we have

$$(x^2+y^2)(x^2+z^2)(y^2+z^2)=\begin{vmatrix}x, & -y\\ y, & x\end{vmatrix}\begin{vmatrix}z, & x\\ -x, & z\end{vmatrix}\begin{vmatrix}y, & z\\ -z, & y\end{vmatrix}$$

$$=\begin{vmatrix}xyz-xy^2-x^2z-yz^2, & -x^2y-xz^2-y^2z+xyz\\ x^2y+xz^2+y^2z-xyz, & xyz-xy^2-x^2z-yz^2\end{vmatrix}$$

$$=(xyz-xy^2-x^2z-yz^2)^2+(x^2y+xz^2+y^2z-xyz)^2.$$

[Other forms can be similarly obtained.—EDITOR].

Also solved by *John Faught, M. A. Gruber, J. Scheffer, and C. D. Schmitt.*

37. Proposed by H. M. CASH, Gibson, Ohio.

The area of the segment of a circle=c, and radius=r. Find height of segment.

Solution by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

Denoting the arc by 2θ, and the height by h, we have for the area of the segment $r^2\theta-r^2\sin\theta\cos\theta=r^2\theta-\frac{1}{2}r^2\sin2\theta$.

$\therefore\quad 2\theta-\sin2\theta=\frac{2c}{r^2}$, which transcendental equation is to be solved to find θ, then h is found by the relation, $h=2r\sin^2\frac{1}{2}\theta$.

We might express the area also by r and h directly, and we would then have the transcendental equation $\cos^{-1}\frac{r-h}{r}-(r-h)\sqrt{(2rh-h^2)}=\frac{c}{r^2}$; which, however, is too inconvenient for solution.

Also solved by *A. H. Bell, J. A. Calderhead, and C. D. Schmitt.*

38. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

A man sold 2 horses and a mule for \$286.90. On the first horse he gained as much per cent. as the horse cost dollars, and gained $\frac{5}{8}$ as much per cent. on the second horse as the first, and he looses \$9.10 on the mule. His net gain was \$86.90. What was the cost and selling price of each?

Solution by T. W. PALMER, M. A., Professor of Mathematics, University of Alabama.

Let x=cost of 1st horse, y=cost of 2nd horse, and z=cost of mule. Then $x+y+z=\$286.90-\$86.90=\$200\ldots.(1)$.

$$\frac{x^2}{100}+\frac{5xy}{800}=\$86.90+\$9.10=\$96\ldots.(2).$$

$$\therefore\quad 8x^2+5xy=76800,\text{ or } y=\frac{8(9600-x^2)}{5x}\ldots.(3).$$

And substituting this in (1), we obtain $z=\frac{3x^2+1000x-76800}{5x}$.

From these two indeterminate equations, we find $x<97$ and $x>65$, and $z>9.10$. An indefinite number of solutions is possible. But in (3), $x=80$ satisfies all the conditions. When $x=80$, $y=64$, and $z=56$.

Also $x+\frac{x^2}{100}=144$, $y+\frac{5xy}{800}=96$, and $z-9.10=46.90$.

$$\therefore\quad \left.\begin{array}{l} x=\$80 \\ y=\$64 \\ z=\$56 \end{array}\right\}=\text{cost};$$

$$\left.\begin{array}{l} x+\frac{x^2}{100}=\$144 \\ y+\frac{5xy}{800}=\$\ 96 \\ z-9.10=\$\ 46.90 \end{array}\right\}=\text{selling price}.$$

Also solved by *J. A. Calderhead, J. Scheffer, C. D. Schmitt, and G. B. M. Zerr.*

A solution of number 35 was received from *Prof. Cooper D. Schmitt*, after last issue of MONTHLY had gone to press.

PROBLEMS.

48. Proposed by SETH PRATT, C. E., Assyria, Michigan.

What is the interest of $500 for 10 years at 10% per annum, when the intervals of compounding are infinitely small?

49. Proposed by P. S. BERG, Apple Creek, Ohio.

A man having lent $6000 at 6% interest, payable quarterly, wishes to receive his interest in equal proportions monthly, and in advance; how much ought he to receive each month?

GEOMETRY.

Conducted by B.F.FINKEL, Kidder, Missouri. All Contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

36. Proposed by O. W. ANTHONY, Mexico, Missouri.

From two points, one on each of the opposite sides of a parallelogram, lines are drawn to the opposite vertices. Through the points of intersection of these lines a line is drawn. Prove that it divides the parallelogram into two equal parts.

II. Solution by LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago.

The line in question IK will cut the parallelogram into two equal parts if it is proved to pass through O, the intersection of the diagonals. The latter theorem is true for any quadrilateral (Pappus, *Mathematicae Collectiones*, VII, p. 139).

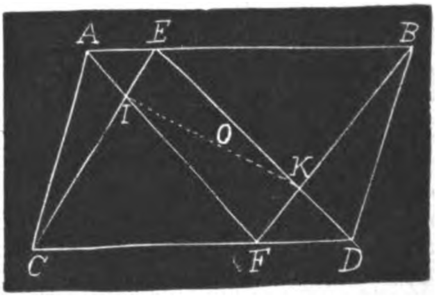


Fig. 1.

If a hexagon $AB'CA'BC'$ has its summits of even order upon one straight line and those of odd order upon another, the three pairs of opposite sides (AB' and $A'B$, BC' and $B'C$, CA' and $C'A$) intersect in three points on a straight line. See Cremona for proof. But even this is only Pascal's theorem for a hexagon inscribed in a conic, when the latter degenerates to a pair of straight lines.

Analytical proof for parallelogram:

Take as axes lines ∥ to AB and BD through O. Let $AB=2b$, $BD=2a$. Call $I(x',y')$ and $K(x_1, y_1)$. Equation to AF is $(x+b)(y'-a)=(y-a)(x'+b)$. Equation to DE is $(x+b)(y'+a)=(y+a)(x'+b)$. Hence the co-ordinates of F are $\left(\frac{2ax'+by'+ab}{a-y'}, -a\right)$; of E, $\left(\frac{2ax'-by'+ab}{a+y'}, a\right)$.

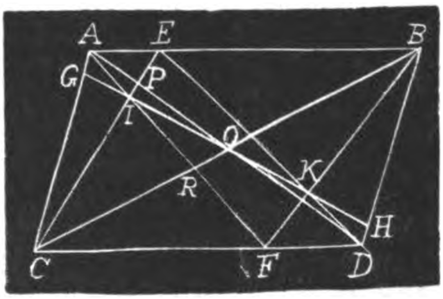
Hence, equation to BF is $a(x-b)(y'-a)=(y-a)(ax'+by')$: equation to ED is $a(x-b)(y'+a)=(y+a)(ax'-by')$.

∴ the co-ordinates of K are $x_1=\frac{a^2x'(x'+b)}{a^2x'+by'^2}$; $y_1=\frac{a^2y'(x'+b)}{a^2x'+by'^2}$.

∴ $\frac{x_1}{y_1}=\frac{x'}{y'}$, or IK passes through the origin O.

III. Solution by J. K. ELLWOOD, A. M., Principal Colfax School. Pittsburg, Pennsylvania.

Let E and F be the points taken in opposite sides of the parallelogram $ABCD$, GH the line drawn through the points of intersection.



FIRST.—The intersection, O, of the diagonals of $ABCD$ is in the line GH.

The △'s CIF and AIE are similar, as are FKD and EKB, and AOB and COD. Hence

$BO:OC::AB:CD$; or $BO\times CD=OC\times AB$....(1).

$FK:BK::FD:EB$; or $FK\times EB=BK\times FD$....(2).

$CI:EI::FI:AI$; by composition,

$CI:CE::FI:AF$; or $CI\times AF=CE\times FI$....(3).

The sides produced of the △'s CIF, AIE, CIR are cut by the lines DPA, COB, and OA, respectively.

∴ $CP\times FD\times AI=CD\times AF\times IP$*....(4).

$IR\times AB\times CE=AR\times EB\times CI$....(5).

$CO\times IP\times AR=CP\times AI\times OR$....(6).

Multiplying equations (1),(2),(3),(4),(5),(6) together, we have

$BO\times FK\times IR=BK\times FI\times OR$....(7).

∴ (By appended proof **) O is in the line $GIKH$.

SECOND.—In the similar △'s AOG and HOD, $AO=OD$.

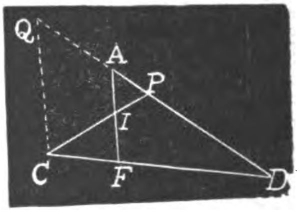
∴ $HD=AG$. ∴ $GC=HB$, and $AG+HB=DH+CG$.

Area trapezoid $AGHB=\frac{1}{2}(AG+HB)\times$alt.

Area trapezoid $GCDH=\frac{1}{2}(DH+CG)\times$alt.

Then, since $AG+HB=DH+CG$, these areas are equal, and the line GH divides $ABCD$ into two equal parts. Q. E. D.

EXPLANATORY PROOF. Let CFI (see fig. 1.) be any $\triangle$, and produce the sides to D, P, A in any *straight line*, as DA.



Through C draw CQ parallel to FA and meeting DA, produced, in Q. The $\triangle$'s DAF and DQC are similar, as are API and QPC.

Hence $FD:AF::CD:CQ$, and

$AI:IP::CQ:CP$.

$\therefore \quad FD\times AI:AF\times IP::CD:CP$.

$\therefore \quad FD\times AI\times CP=AF\times IP\times CD$.

This is equation (4)a bove; (5) and (6) may be similarly obtained.

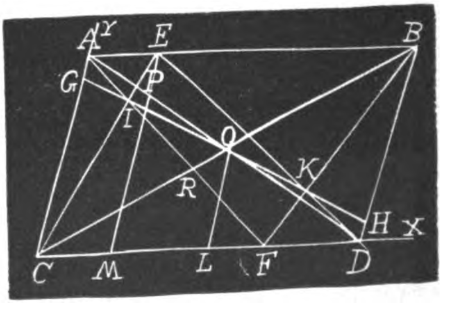
It may be shown in a similar manner that, if a *straight line* cuts two sides and the third side produced of a $\triangle$, the product of any three of the non-adjacent segments (of the sides) is equal to the product of the other three segments. The produced side is one segment, the prolongment another.

** CONVERSELY. If three points divide the two sides and determine the prolongment of the third side produced so that the product of any three non-adjacent segments shall be equal to the product of the other three, then are these points in the same *straight line*.

Equation (7) is derived from the $\triangle$ BFR. The points K, O are in the *sides*, and I is in RF produced. Since in equation (7) the product of three non-adjacent segments is equal to the product of the other three, the three points I, O, K are in the straight line GH.

IV. **Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.**

Let E, F be the points. I, K the intersection of ED, FB and EC, FA, O the intersection of the diagonals BC, AD. Let $CD=AB=a$, $AC=ME=BD=b$, $CM=c$, $CF=d$, h=the perpendicular from A on BD.



Then $CL=\frac{1}{2}a$, $LO=\frac{1}{2}b$,

$\frac{y}{x}=\frac{b}{c}$, equation to CE; $\frac{x}{d}+\frac{y}{b}=1$, equation to AF. $y=\frac{b(a-x)}{a-c}$, equation to DE, $y=\frac{b(x-d)}{a-d}$, equation to BF.

$\left(\frac{cd}{c+d}, \frac{bd}{c+d}\right)$ co-ordinates of I, $\left(\frac{a^2-dc}{2a-c-d}, \frac{b(a-d)}{2a-c-d}\right)$ co-ordinates of K.

$(2cd-ac-ad)y-(bd-bc)x=bd(c-a)$, the equation to IO.

This line cuts DE at the point $\left(\frac{a^2-dc}{2a-c-d}, \frac{b(a-d)}{2a-c-d}\right)$

$\therefore$ IO passes through K, and IO and GH are the same line.

In the triangle GOA and HKD, $AO=OD$, $\angle AOG=\angle DOH$,

$\angle OAG = \angle ODH$. $\therefore$ $GA = DH$. Similarly $CG = BH$.

$\therefore$ $AG + BH = CG + DH$.

$\therefore$ $\frac{1}{2}h(AG + BH) = \frac{1}{2}h(CG + DH)$.

$\therefore$ Area $AGHD$ = area $CGHD$.

Good solutions of this problem were received from *Professors Wm. Symmonds, and Cooper D. Schmitt.*

This problem has proved to be a very interesting one and for that reason we have given it extra space. The proposition to which Prof. Ellwood has given a proof explanatory to the proposition under consideration is known as the proposition of *Menelaus*. See *Halsted's Elementary Synthetic Geometry*, p. 117. Editor.

PROBLEMS.

42. Proposed by G. I. HOPKINS, Instructor in Mathematics and Physics in High School, Manchester, New Hampshire.

If the bisectors of two angles of a triangle are equal the triangle is isosceles.

[The term *bisector* in this theorem means the line which divides an angle into two equal parts and terminates in the opposite side.]

43. Proposed by J. F. W. SCHEFFER, Hagerstown, Maryland.

The consecutive sides of a quadrilateral are a, b, c, d. Supposing its diagonals to be equal, find them and also the area of the quadrilateral.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

27. Proposed by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

A runs around the circumference of a circular field with velocity m feet; B starts from the centre with velocity $n > m$ feet to catch A. The straight line joining their positions always passes through the centre. Find the equation to the curve described by B, the distance he runs and the time occupied.

I. Solution by A. H. HOLMES, Brunswick, Maine, H. W. DRAUGHON, Olio, Mississippi, and the PROPOSER.

Let A be the point of starting of the pursued. P, B, the position of the pursuer and pursued at any time.

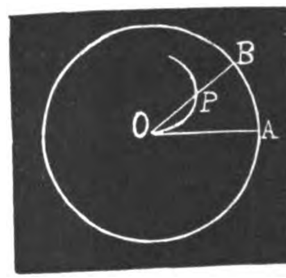
Let $OA = a$, $OP = r$, $\angle BOA = \theta$, $\dfrac{m}{n} = u$, arc $OP = s$.

Then $us=a\theta$....(1). $\therefore$ $s=\frac{a\theta}{u}=\frac{an\theta}{m}$ is the intrinsic equation to the curve.

From (1) $\frac{ds}{d\theta}=\frac{a}{u}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$.

$$\therefore\ r^2+\left(\frac{dr}{d\theta}\right)^2=\frac{a^2}{u^2}\ \text{and}\ d\theta=\frac{dr}{\sqrt{\left(\frac{a^2}{u^2}-r^2\right)}}.$$

$\therefore$ $\theta=\sin^{-1}\frac{ur}{a}$....(2).



$\therefore$ $r=\frac{a}{u}\sin\theta=\frac{an}{m}\sin\theta$, is the polar equation and $m(x^2+y^2)=any$, is the rectangular equation.

The value of θ from (2) in (1) gives

$$s=\frac{a}{u}\sin^{-1}\frac{ur}{a},$$

the length for any value of r. When $r=a$,

$$s=\frac{a}{u}\sin^{-1}u=\frac{an}{m}\sin^{-1}\frac{m}{n}=\text{distance run}.$$

$$t=\text{time}=\frac{s}{n}=\frac{a}{m}\sin^{-1}\frac{m}{n}.$$

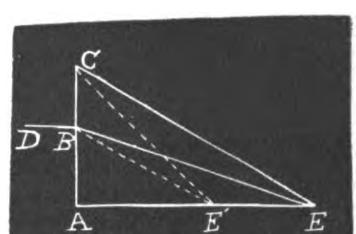
Also solved by *Professors O. W. Anthony, J. Scheffer, and William Symmonds.*

28. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

How far from the stage must Miss Love sit in order that she *may see to best advantage* Mr. Rich deliver the valedictory oration?

Solution by O. W. ANTHONY, Missouri Military Academy, Mexico, Missouri, and the PROPOSER.

Let E represent the position of Miss Love's eyes; DB the stage from which Mr. Rich orates; AB, $=m$ feet, the height of the stage above Miss Love's eyes; BC, $=n$ feet, the height of Mr. Rich; and AE, $=x$ feet, the required distance. In order that Miss Love may *see* Mr. Rich *to best advantage*, the angle BEC must be a maximum, that is,



$$U=\tan^{-1}\left(\frac{m+n}{x}\right)-\tan^{-1}\left(\frac{m}{x}\right)=\text{a Maximum}.$$

$$\therefore\ \frac{dU}{dx}=\frac{m}{x^2+m^2}-\frac{m+n}{x^2+(m+n)^2}=0....(1).$$

Whence $x=\sqrt{[m(m+n)]}$ feet, which is the required distance.

29. Proposed by CHARLES E. MYERS, Canton, Ohio.

A hen running at the rate of $n=2$ feet per second, on the circumference of a circle, radius $r=50$ feet, is observed by a hawk $a=600$ feet directly above the center.

The hawk at once starts in pursuit, flying at the rate of $m=5$ feet per second and keeping always in a straight line with the starting point and the hen.

Determine the path followed and the distance the hawk will fly before catching the hen.

Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let the origin be at the vertex of the cone around which the path of the hawk winds. σ=length of the hawk's path, s=the length of the projection of this path on the plane (x,y), ρ=radius vector of this projection, $\frac{n}{m}=u$, $\frac{r}{a}=c$. Then $x^2+y^2=c^2z^2$ is the equation of the cone, also $u\sigma=r\theta$, where θ is the angle subtended by the hen's path at the centre of the circle.

$$\therefore\quad d\sigma=\frac{r}{u}\,d\theta=\sqrt{dx^2+dy^2+dz^2}=\sqrt{ds^2+dz^2}$$

$$=\left\{\rho^2+\left(\frac{d\rho}{d\theta}\right)^2+\left(\frac{dz}{d\theta}\right)^2\right\}^{\frac{1}{2}}d\theta,\text{ but }\rho^2=x^2+y^2=c^2z^2.\quad\therefore\ \rho=cz.\quad\therefore\ dz=\frac{d\rho}{c}.$$

$$\therefore\quad\frac{d\sigma}{d\theta}=\frac{r}{u}=\left\{\rho^2+\left(\frac{d\rho}{d\theta}\right)^2+\frac{1}{c^2}\left(\frac{d\rho}{d\theta}\right)^2\right\}^{\frac{1}{2}}.$$

$$\therefore\quad\frac{d\theta}{d\rho}=\frac{\sqrt{c^2+1}}{c}\cdot\frac{1}{\left\{\frac{r^2}{u^2}-\rho^2\right\}^{\frac{1}{2}}}.\quad\theta=\frac{\sqrt{c^2+1}}{c}\sin^{-1}\frac{u\rho}{r}.$$

$$\therefore\quad\theta=\frac{\sqrt{r^2+a^2}}{r}\sin^{-1}\left(\frac{n\rho}{mr}\right);\ \rho=\frac{mr}{n}\sin\frac{r\theta}{\sqrt{r^2+a^2}}.$$

$$z=\rho/c=\frac{am}{n}\sin\frac{r\theta}{\sqrt{r^2+a^2}}=1500\sin.\frac{\theta}{\sqrt{145}}.$$

$$x=\rho\cos\theta=\frac{mr}{n}\sin\frac{r\theta}{\sqrt{r^2+a^2}}\cos\theta=125\sin\frac{\theta}{\sqrt{145}}\cos\theta.$$

$$y=\rho\sin\theta=\frac{mr}{n}\sin\frac{r\theta}{\sqrt{r^2+a^2}}\sin\theta=125\sin\frac{\theta}{\sqrt{145}}\sin\theta.$$

These values of x, y, z determine the hawk's path.

$$\text{Also } u\sigma=r\theta.\quad\therefore\ \sigma=\frac{mr}{n}\left[\frac{\sqrt{r^2+a^2}}{r}\sin^{-1}\left(\frac{n\rho}{mr}\right)\right]_0^r=\frac{m\sqrt{r^2+a^2}}{n}\sin^{-1}\left(\frac{n}{m}\right).$$

$$\therefore\quad\sigma=125\sqrt{145}\sin^{-1}\left(\frac{2}{5}\right)=619.406\text{ feet approximately, the distance}$$

the hawk flies before catching the hen.

Solved in a similar manner by *Professor J. F. W. Scheffer*.

PROBLEMS.

38. Proposed by L. B. FILLMAN, St. Petersburg, Pennsylvania.

The diameter of the circular base of a dome is $10=a$ feet, which is also the distance from any point on the circumference of the base to any point on the opposite side of the dome from base to apex. Find volume of dome. [See Prob. 21. --EDITOR.]

39. Proposed by J. C. GREGG, Brazil, Indiana.

Show that the curve

$$x=9a\sin\theta-4a\sin^3\theta$$
$$y=-3a\cos\theta+4a\cos^3\theta$$

is symmetrical to the axes, and has double points and cusps: find the lengths of the arcs, (a) between the double points, (b) between a double point and a cusp, (c) and the arc connecting two cusps and not passing through a double point. [*Rice and Johnson's Integ. Cal.* (abridged), p. 176.]

MECHANICS.

Conducted by B.F.FINKEL, Kidder, Missouri. All Contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

15. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Show that the *eastward deviation* of bodies falling from a great height is

$$E_d=\frac{4\pi t(H-\frac{1}{2}\triangle)\cos\phi}{3T}.$$

II. Solution by Professor G. B. M. ZERR, A. M., Principal of High School Staunton, Virginia.

Let OZ be the axis of z, OX, the axis x, OY, the axis of y.

Let ϕ=the latitude of O,

β=angular velocity of the earth around DO,

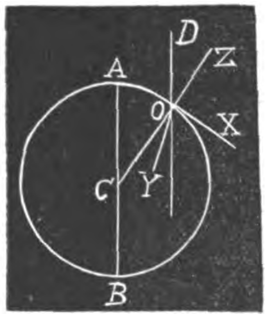
$\triangle$=excess of descent in vacuo over that of air,

T=time in seconds of siderial day,

H=height of body above the earth.

Also let OX be tangent to the meridian and OY perpendicular to it, and their positive directions respectively south and west. The velocity of the body eastward at the moment it is dropped from $z_1=\beta H\cos\phi$. Now if gravity did not alter its direction, owing to the rotation of the earth, the body would describe a parabola and the easterly deviation would be $(\beta H\cos\phi)t$, where t=time of falling. But the rotation β about OD is equivalent to $\beta\sin\phi$ about

OZ, and $\beta \cos\phi$ about OX. The former does not alter the position of OC, while the latter turns OC in a time t through an angle $\beta \cos\phi t$. Hence the body is acted upon by a westerly component, due to the change of direction of gravity, $=g \sin(\beta \cos\phi t)=g\, \beta \cos\phi t$, since βt is small. Now let y be the distance the body is in space from the plane XZ at the moment the body begins to fall, and then the equation of motion of the body in space is $\frac{d^2y}{dt^2}=g\,\beta\, t \cos\phi$. Integrating this, and remembering that, as explained above, $\frac{dy}{dt}=-\beta H \cos\phi$, we get $y=-\beta H\, t \cos\phi+\frac{1}{6}\, g\beta\, t^3 \cos\phi$. $\therefore$ $E_d=\beta\, t \cos\phi(H-\frac{1}{6}\, gt^2)$. But $\frac{1}{2}\, gt^2=H+\triangle$ and the centrifugal force at the equator $=\beta^2 r=\frac{4\pi^2 r}{T^2}$, where r=radius of the earth.



$$\therefore \quad \beta^2=\frac{4\pi^2}{T^2} \text{ and } \beta=\frac{2\pi}{T}. \text{ Substituting we get}$$

$$E_d=\frac{2\pi t \cos\phi}{T}\left\{H-\tfrac{1}{3}(H+\triangle)\right\}=\frac{2\pi t \cos\phi}{T}(\tfrac{2}{3}H-\tfrac{1}{3}\triangle)$$

$$=\frac{2\pi t \cos\phi}{3T}(2H-\triangle)=\frac{4\pi t(H-\frac{1}{2}\triangle) \cos\phi}{3T}.$$

[NOTE.--No solution has yet been received to problem 17. EDITOR.]

PROBLEMS.

24. Proposed by Professor J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

A sphere whose center of gravity does not coincide with its geometrical center is placed on a rough inclined plane. State under what circumstances the sphere will slide without rolling, roll without sliding, and neither roll nor slide.

25. Proposed by Professor GEORGE LILLEY, LL. D., Ex-President of Washington State Agricultural College and School of Science, Portland, Oregon.

It is known that if the velocity of a certain freight train is 30 miles an hour it can be brought to a stand still in a distance of 500 feet by setting the brakes. It was stopped in 1200 feet by setting the brakes. Find its velocity, the forces exerted by the brakes being the same in each case.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

18. Proposed by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Decompose into the sum of two squares the number $17^3.73^5$.

I. Solution by the PROPOSER.

$(a^2+b^2)(a^2+b^2)=(a^2\pm b^2)^2+(ab\mp ab)^2=A^2+B^2.$

$(a^2+b^2)^3=(A^2+B^2)(a^2+b^2)=(Aa\pm Bb)^2+(Ab\mp Ba)^2.$

$\therefore\quad (a^2+b^2)^3=\{a(a^2-3b^2)\}^2+\{b(3a^2-b^2)\}^2,$

$=\{a(a^2+b^2)\}^2+\{b(a^2+b^2)\}^2.$

Similarly $(c^2+d^2)^5=\{c(c^4-10c^2d^2+5d^4)\}^2+\{d(5c^4-10c^2d^2+d^4)\}^2,$

$=\{c(c^2+d^2)^2\}^2+\{d(c^2+d^2)^2\}^2,$

$=\{c(c^2+d^2)(3d^2-c^2)\}^2+d(c^2+d^2)(3c^2-d^2)\}^2.$

Let $a=4$, $b=1$, $c=8$, $d=3$. $\therefore\quad 17^3=52^2+47^2=68^2+17^2.$

$73^5=10072^2+44403^2=42632^2+15987^2=21608^2+40077^2.$

$\therefore\quad 17^3.73^5=3092572^2+788035^2=2357900^2+2150653^2$

$=362372^2+3170755^2=1811860^2+2627197^2$

$=3190628^2+69955^2=2848180^2+1439747^2$

$=3099580^2+560003^2=1068428^2+3007235^2$

$=2835028^2+1465475^2=1172380^2+2968253^2$

$=2782340^2+1565197^2=1835572^2+2612685^2.$

$\therefore$ the sum of two squares in twelve ways.

II. Solution by H. W. DRAUGHON, Olio, Mississippi, and O. W. ANTHONY, Missouri Military Academy, Mexico, Missouri.

Since $17^3\times73^5=17^2\times73^4\times17\times73=17^2\times73^4\times1241.$

And $1241=35^2+4^2=29^2+20^2$. Therefore,

$17^3\times73^5=(17\times73^2\times35)^2+(17\times73^2\times4)^2=(17\times73^2\times29)^2+(17\times73^2\times20)^2.$

III. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

Solution by determenants,

$$17=\begin{vmatrix}4 & -1\\ 1 & 4\end{vmatrix},\quad 73=\begin{vmatrix}8 & -3\\ 3 & 8\end{vmatrix}\quad 17^2=\begin{vmatrix}17 & 0\\ 0 & 17\end{vmatrix}\quad 73^4=\begin{vmatrix}5329 & 0\\ 0 & 5329\end{vmatrix}$$

$$17^3=\begin{vmatrix}17 & 0\\ 0 & 17\end{vmatrix}\times\begin{vmatrix}4 & -1\\ 1 & 4\end{vmatrix}=\begin{vmatrix}68 & 17\\ -17 & 68\end{vmatrix}$$

$$17^3\times73=\begin{vmatrix}68 & 17\\ -17 & 68\end{vmatrix}\times\begin{vmatrix}8 & -3\\ 3 & 8\end{vmatrix}=\begin{vmatrix}344-51 & 204+136\\ -136-204 & -51+344\end{vmatrix}=\begin{vmatrix}493 & 340\\ -340 & 493\end{vmatrix}.$$

$$\text{Then } 17^3\times 73^5=\begin{vmatrix}493 & 340\\ -340 & 493\end{vmatrix}\times\begin{vmatrix}5329 & 0\\ 0 & 5329\end{vmatrix}=(493^2+340^2)(5329)^2$$

$=(73)^2(493)^2+(73)^2(340)^2$, which is one set of answers.

Also solved by *R. J. Adcock*, and *M. A. Gruber*.

19. Proposed by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C

Find three positive integer numbers whose sum is a cube, and, also, the sum of any two diminished by the third a cube.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Let x, y, and z=the three positive integers.

$$\text{Then}\quad \begin{aligned} x+y+z&=a^3\\ x+y-z&=b^3\\ x+z-y&=c^3\\ y+z-x&=d^3\end{aligned}$$

$$\text{Whence}\quad x+y+z=b^3+c^3+d^3=a^3;$$

$$x=\frac{b^3+c^3}{2};\ y=\frac{b^3+d^3}{2};\ z=\frac{c^3+d^3}{2}.$$

This is a problem in which the sum of three cubes=a cube. Take $3^3+4^3+5^3=6^3$. But as the numbers are to be integers, we multiply by 2, and obtain $6^3+8^3+10^3=12^3$. $\therefore\ x=\dfrac{6^3+8^3}{2}=364$; $y=\frac{1}{2}(6^3+10^3)=608$; and $z=\frac{1}{2}(8^3+10^3)=756$. The number of answers is infinite.

II. Solution by R. J. ADCOCK, Larchland, Warren County, Illinois.

Let x, y, z, be the three numbers; then $x+y+z=u^3$, $x+y-z=r^3$, $x+z-y=v^3$, $z+y-x=s^3$, by conditions.

$$\text{Wherefore}\quad x+y+l=u^3=r^3+v^3+s^3,\ x=\frac{r^3+v^3}{2},\ y=\tfrac{1}{2}(r^3+s^3),$$

$z=\frac{1}{2}(v^3+s^3)$. The most general equation yet obtained by me for the sum of three cubes=a cube, is found from,

$[(ax^3+dy^3)x]^3+[(bx^3+hy^3)y]^3+[(cx^3-hy^3)y]^3=[(ax^3+gy^3)x]^3$, by expanding, equating coefficients of similar terms with respect to x and y, eliminating d, g, and h, giving the identical equation,

$$[9a^3bx^3y+(b^2-bc+c^2)^2y^4]^3+[9a^3cx^3y-(b^2-bc+c^2)^2y^4]^3$$
$$+[9a^4x^4-3(b^2-bc+c^2)acxy^3]^3=[9a^4x^3+3ab(b^2-bc+c^2)xy^3]^3.$$

By numbers for the letters in the above, some of the resulting equations are $3^3+4^3+5^3=6^3$, $1^3+6^3+8^3=9^3$, $3^3+10^3+18^3=19^3$,

$7^3+14^3+17^3=20^3$, $4^3+17^3+22^3=25^3$, $11^3+15^3+27^3=29^3$.

Then the three positive integer numbers are $x=\frac{11^3+15^3}{2}$, $y=\frac{11^3+27^3}{2}$.

$z=\frac{15^3+27^3}{2}$. Also x, y, z may be found from any equation, including the algebraicsum, for the sum of three cubes=a cube, by first multiplying each cube by 2^3.

Also solved by *O. W. Anthony, H. W. Draughon, C. D. Schmitt, and G. B. M. Zerr.*

PROBLEMS.

27. **Proposed by J. W. NICHOLSON, LL. D., President and Professor of Mathematics, Louisiana State University and A. and M. College, Baton Rouge, Louisiana.**

Required a formula for finding five integers the sum of whose cubes is a cube.

28. **Proposed by DAVID E. SMITH, Ph. D., Professor of Mathematics, Michigan State Normal School, Ypsilanti, Michigan.**

Decompose the product 97.673.257 into the sum of two fourth powers.

AVERAGE AND PROBABILITY.

Conducted by B.F.FINKEL, Kidder, Missouri. All Contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

13. **Proposed by I. L. BEVERAGE, Monterey, Virginia.**

Find the mean values of the roots of the quadratic $x^2-ax+b=0$, the roots being known to be real, but b being unknown and positive.

Solution by P. S. BERG, Apple Creek, Ohio, and JOHN DOLMAN, Jr., Counsellor-at-law, Philadelphia, Penn., and J. M COLAW, A. M., Principal of High School, Monterey, Virginia.

Solving the given equation, $x=\frac{1}{2}a\pm\sqrt{(\frac{1}{4}a^2-b)}$.

Therefore, if b be positive and x real, b cannot exceed $\frac{1}{4}a^2$. If β be the smaller of the two roots, its mean value, therefore, is

$$(1\div\tfrac{1}{4}a^2)\int_0^{\frac{1}{4}a^2}\beta db=\frac{4}{a^2}\int_0^{\frac{1}{4}a^2}(\tfrac{1}{2}a+\surd(\tfrac{1}{4}a^2-b)db=\frac{4}{a^2}[\tfrac{1}{2}ab]_0^{\frac{1}{4}a^2}+\frac{4}{a^2}[\tfrac{1}{12}\surd(a^2-4b)^3]_0^{\frac{1}{4}a^2}$$

$$-\frac{a}{3}=\frac{a}{2}-\frac{a}{3}=\frac{1}{6}a.$$

The mean value of the larger root is, therefore, $\frac{5}{6}a$.

Also solved in a similar manner by *Professors Matz, Zerr, and Draughon.*

14. Proposed by CHARLES E. MYERS, Canton, Ohio.

$\frac{1}{3}$ of all the melons in a patch are not ripe, and $\frac{1}{4}$ of all the melons in the same patch are rotten, the remainder being good. If a man enters the patch on a dark night and takes a melon at random, what is the probability that he will get a good one?

Solution by H. W. DRAUGHON, Olio, Mississippi, and G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let $12n$ = the whole number of melons in the patch. Then $4n$ are not ripe and $3n$ are rotten. The $3n$ rotten melons may be included in the $4n$ not ripe melons in which case there would be $8n$ good melons, or the $3n$ rotten may not be included in the $4n$ not ripe melons in which case there would be $12n-(3n+4n)=5n$ good melons.

$\therefore$ there cannot be less than $5n$ nor more than $8n$ good melons.

$\therefore$ the chance of a good one $=\frac{1}{2}\left(\frac{5n+8n}{12n}\right)=\frac{13}{24}$.

The chance of a not ripe one $=\frac{1}{2}\left(\frac{n+4n}{12n}\right)=\frac{5}{24}$.

The chance of a rotten one $=\frac{1}{2}\left(\frac{0+3n}{12n}\right)=\frac{1}{8}$.

The chance of a not ripe and rotten one $=\frac{1}{2}\left(\frac{0+3n}{12n}\right)=\frac{1}{8}$.

$\therefore$ $\frac{13}{24}+\frac{5}{24}+\frac{1}{8}+\frac{1}{8}=1$ as it should be.

Solutions o this problem were received from *P. S. Berg, F. P. Matz, J. M. Colaw.*

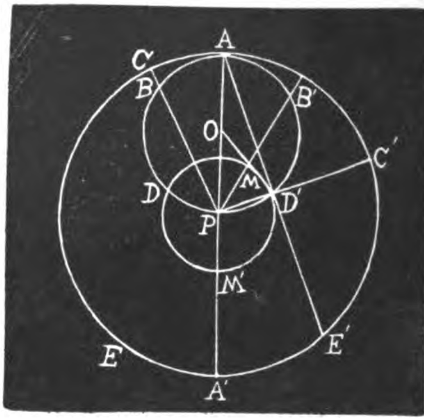
15. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Todhunter proposes: "From a point in the circumference of a circular field a projectile is thrown at random with a given velocity, which is such that the diameter of the field is equal to the greatest range of the projectile; prove the chance of its falling within the field, is $C=2^{-1}-2\pi^{-1}(\surd2-1), =.236+$." Is this result perfectly correct as to fact?

First Solution by the PROPOSER.

Let P be the point from which the projectile is thrown, $AP=2a$, and $\angle APB=\theta$. Now, if ϕ = the angle of elevation at which the projectile is thrown, and C' = the chance for any given value of θ; then, evidently, the required chance becomes

$$C=\int_0^{\frac{1}{2}\pi} C' d\theta \div \int_0^{\pi} d\theta = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} C' d\theta \ldots (1).$$



Since the range is

$$PB=2a\sin 2\phi=2a\cos\theta,$$

$\sin\phi=\frac{1}{2}[\surd(1+\cos\theta)+\surd(1-\cos\theta)],=R_1$, and $\sin\phi=\frac{1}{2}[\surd(1+\cos\theta)-\surd(1-\cos\theta)]$, $=R_2$.

For all values of $\sin\phi$ *less* than R_2 and *greater* than R_1, the projectile will fall into the field. The whole number of different directions of projection is proportional to the surface S_1 of the hemisphere center of which is P and radius $\frac{1}{2}PB=a\cos\theta$; and this surface is

$$S_1=2\pi a^2\cos^2\theta \ldots (2).$$

The whole number of different directions of projection producing a range *greater* than PB is proportional to the surface S_2 of the zone included between two horizontal planes at the distances $R_1 a\cos\theta$ and $R_2 a\cos\theta$ from the center of the base of the hemisphere; and this surface is

$$S_2=a\cos\theta\surd(1-\cos\theta)\times 2\pi a\cos\theta=2\pi a^2\cos^2\theta\surd(1-\cos\theta)\ldots (3).$$

That is, the whole number of different directions of projection giving a range *less* than PB is proportional to S_1-S_2; and, therefore, we have the chance for any given value of θ

$$C'=\frac{S_1-S_2}{S_1}=1-\frac{S_2}{S_1}=1-\surd(1-\cos\theta)=1-\surd(2)\sin\tfrac{1}{2}\theta \ldots (4).$$

$$\therefore\quad C=\frac{1}{\pi}\int_0^{\frac{1}{2}\pi} C' d\theta=\frac{1}{\pi}\int_0^{\frac{1}{2}\pi}[1-\surd(2)\sin\tfrac{1}{2}\theta]d\theta=\frac{1}{2}-2\left(\frac{\surd 2-1}{\pi}\right)$$

$$=2^{-1}-2\pi^{-1}(\surd 2-1),=.236+,$$ which is Todhunter's result.

Second Solution.

Obviously the number of *favorable* chances is represented by the area of the circle $ABDPD'B'A-O$, and the total number of chances is represented by the area of the circle $ACEA'E'C'A-P$. Therefore, the required chance is $C=\frac{1}{4},=.25$.

Third Solution.

For any range PM the number of *favorable* chances is represented by the area $\mathbf{A}'$ of the double segment $DMD'P$, and the total number of chances is represented by the area of the circle $DM'D'M-P$. Let $\angle PAD'=\omega$; then $\angle POD'=2\omega$, $\angle MPD'=(90-\omega)$, $PM=PD'=2a\sin\omega$, and $\mathbf{A}'=2$(Sector MPD'+Sector POD'−Triangle $OD'P$)$=2a^2[\pi\sin^2\omega$ $-(\omega+\sin\omega\cos\omega-2\omega\cos^2\omega)]$. Hence the required chance becomes

$$C=2a^2\int_0^{\frac{1}{2}\pi}\mathbf{A}'d\omega\div 4\pi a^2\int_0^{\frac{1}{2}\pi}\sin^2\omega d\omega=\frac{1}{2}-\frac{2}{\pi^2},=.297+.$$

Fourth Solution.

From A draw at random the chord AE', put $\angle A'AE'=\Psi$; then $AD'=2a\cos\Psi=r_1$, and $AE'=4a\cos\Psi=r_2$. For any value of Ψ the number of *favorable* chances is represented by the *sectoral* surface PAD', and the total number of chances by the sectoral surface $A'AE'$. The chance in consideration, therefore, becomes

$$C=2\int_0^{\frac{1}{2}\pi}\int_0^{r_1} d\Psi r dr \div 2\int_0^{\frac{1}{2}\pi}\int_0^{r_2} d\Psi r dr=\frac{1}{4},=.25.$$

Fifth Solution.

Put $\angle PAD'=\omega$, then $\angle APD'=(\frac{1}{2}\pi-\omega)$. Therefore, for any range PD' the projectiles falling on the circular arc DMD' are within the field. Consequently the required chance becomes

$$C=4a\int_0^{\frac{1}{2}\pi}(\tfrac{1}{2}\pi-\omega)\sin\omega d\omega \div 4\pi a\int_0^{\frac{1}{2}\pi}\sin\omega d\omega=\frac{1}{2}-\frac{1}{\pi},=.182.$$

Sixth Solution.

The number of *favorable* chances is proportional to $2\angle MPD'=2(\frac{1}{2}\pi-\Psi)$, and the total number of chances is proportional to $2(\pi)$. Hence the required chance becomes

$$C=2\int_0^{\frac{1}{2}\pi}(\tfrac{1}{2}\pi-\Psi)d\Psi \div 2\pi\int_0^{\frac{1}{2}\pi}d\Psi=\tfrac{1}{4},=.25.$$

NOTE—Since the projectiles are *thrown* at random, they should *fall* at random; and, therefore, the required chance should be $C=\frac{1}{4},=.25$. To interpret *this* result is apparently easy enough; but to interpret Todhunter's result, or the results $C=.297+$ and $C=.182$, is not so easy. In fact, the interpretation of these three results becomes all the more remarkable when we note that their average value $C_A=.238+$, which average value differs but slightly from Todhunter's result.

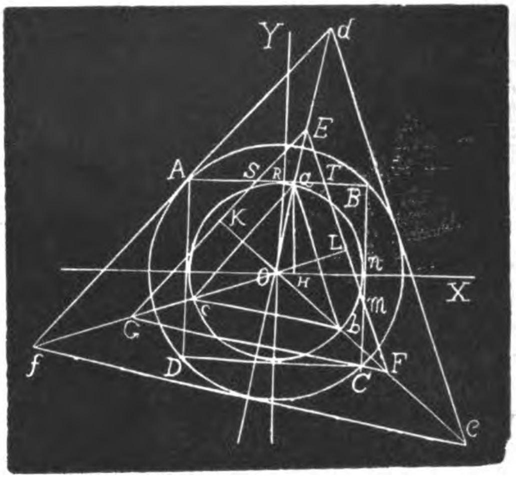
This problem was also solved by G. B. M. Zerr, J. M. Colaw, and John Dolman, Jr., their result agreeing with that given by Todhunter and the first solution of Professor Matz. Professor Zerr says this result is perfectly true as to mathematical fact. We published all of Professor Matz's solutions for comparison.—EDITOR.

18. Proposed by B. F. FINKEL, A. M., Professor of Mathematics in Kidder Institute, Kidder, Missouri.

What is the average volume common to a cube and a rectangular solid one inch square, the axis of rectangular solid being equal to and coinciding with the diagonal of the cube?

Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let $ABCD$ be a section of the rectangular solid, EFG a section of the cube both projected upon the same plane, perpendicular to the diagonal of the cube. The triangle varies in size from abc, the inscribed triangle of the inscribed circle of the square, to def the circumscribed triangle of the circum-circle of the square. Before the triangle is equal in area to abc, all the cube is common to both, after the triangle is greater in area than def, all the rectangular solid is common to both.



Let a=edge of cube, c= side of square $ABCD$. Also suppose $c < \frac{a}{\sqrt{3}}$; for if $c > \frac{a}{\sqrt{3}}$ the section of the cube will be both a triangle and a hexagon.

Let x=side of triangle, p=perpendicular from corner of cube to triangle EFG. Then $p=\frac{x}{\sqrt{6}}$ when $x=ab=\frac{c\sqrt{3}}{2}$, $p=\frac{c}{2\sqrt{2}}$, area $abc=\frac{3\sqrt{3}c^2}{16}$

Volume of pyramid common to both $=\frac{1}{3}p\times abc=\frac{c^3\sqrt{3}}{32\sqrt{2}}$.

When $x=de=c\sqrt{6}$, $p=c$.

$\therefore$ volume rectangular solid common to both$=c^2(\frac{1}{2}a\sqrt{3}-c)=\frac{1}{2}c^2(a\sqrt{3}-2c)$.

$\therefore$ the constant volume common to both solids is double the two volumes just found as we have considered but half the cube.

$$\therefore\ V_1=2\left\{\frac{c^3\sqrt{3}}{32\sqrt{2}}+\frac{1}{2}c^2(a\sqrt{3}-2c)\right\}=\frac{c^2}{16\sqrt{2}}\left\{c\sqrt{3}+16a\sqrt{6}-32c\sqrt{2}\right\}.$$

To find the average volume common to both, that varies, let OX, OY be the axis of reference, $\angle EOY=\theta$.

This volume has sections ranging from the triangle through the quadrilateral, pentagon, hexagon, heptagon, and back through its variations to the square. To find this volume I shall divide its altitude into three equal parts and find the average area of sections passing through these points of division, and then apply the formula of approximate cubature.

For the first point of division $x=\frac{c\sqrt{3}}{3}(\sqrt{2}+1)$.

For the second point of division $x=\frac{c\sqrt{3}}{6}(4\sqrt{2}+1)$.

$OL=OK=\frac{x\sqrt{3}}{6}$, $EH=\frac{x\sqrt{3}}{3}\cos\theta$.

Equation to EF is $x_1 \cos\left(\frac{\pi}{6}-\theta\right)+y_1 \sin\left(\frac{\pi}{6}-\theta\right)=\frac{x_1\surd 3}{6}$.

Equation to EG is $x_1 \cos\left(\frac{5\pi}{6}-\theta\right)+y_1 \sin\left(\frac{5\pi}{6}-\theta\right)=\frac{x_1\surd 3}{6}$.

$RT=\frac{2x\surd 3-3c\cos\theta+3c\surd(3)\sin\theta}{6[\surd(3)\cos\theta+\sin\theta]}$, $RS=\frac{2x\surd 3-3c\cos\theta-3c\surd(3)\sin\theta}{6[\sin\theta-\surd(3)\cos\theta]}$.

$nm=\frac{2x\surd 3-3c\surd(3)\cos\theta-3c\sin\theta}{6[\cos\theta-\surd(3)\sin\theta]}$.

When $x=\frac{c\surd 3}{3}(\surd 2+1)$ the triangle EFG is greater than the inscribed triangle of circum-circle of the square $ABCD$. Hence three times the average area of EST subtracted from the area EFG gives the average area of the section required. Call this area A_2. From $\theta=0$ to $\theta=\theta_1$, area $EST=\frac{1}{2}(RT-RS)(EH-\frac{1}{2}c)$; from $\theta=\theta_1$ to $\theta=\frac{\pi}{4}$, the area $=EST+TBM$ $=\frac{1}{2}(RT-RS)(EH-\frac{1}{2}c)+\frac{1}{2}(RT-\frac{1}{2}c)(\frac{1}{2}c-nm)$. θ_1 is determined from the equation $\frac{2x\surd 3-3c\cos\theta+3c\surd(3)\sin\theta}{6[\sin\theta+\surd(3)\cos\theta]}=\frac{1}{2}c$.

When $x=\frac{c\surd 3}{6}(4\surd 2+1)$, the triangle EFG is greater than the circumscribed triangle of the in-circle of the square $ABCD$. Hence, three times the average area of the triangle TBM subtracted from the area of the square $ABCD$ gives the average area of the section required. Call this area A_3.

Area $TBM=\frac{1}{2}(\frac{1}{2}c-nm)(\frac{1}{2}c-RT)$. Limits of θ are, $\theta=\theta_2$ to $\theta=\frac{\pi}{4}$, where θ_2 is found from the equation, $\frac{2x\surd 3-3c\surd(3)\cos\theta-3c\sin\theta}{6[\cos\theta-\surd(3)\sin\theta]}=\frac{1}{2}c$.

$$\therefore \quad V_2=2x\frac{h}{8}(A_1+3A_2+3A_3+A_4).$$

where $h=c-\frac{c}{2\surd 2}=\frac{c}{2\surd 2}(2\surd 2-1)$. $A_1=$area $abc=\frac{3\surd(3)c^2}{16}$. $A_4=$area $ABCD=c^2$. $\therefore$ $V_2=\frac{3c}{8\surd 2}(2\surd 2-1)(A_2+A_3)+\frac{c}{8\surd 2}(2\surd 2-1)\left(\frac{3\surd(3)c^2}{16}+c^2\right)=\frac{3c}{8\surd 2}(2\surd 2-1)(A_2+A_3)+\frac{c^3}{128\surd 2}(6\surd 6+32\surd 2-3\surd 3-16)$.

$V=V_1+V_2=\frac{3c}{8\surd 2}(2\surd 2-1)(A_2+A_3)+\frac{c^3}{128\surd 2}$ $(6\surd 6+5\surd 3-224\surd 2-16)+ac^2\surd 3$, where A_2 and A_3 are found as indicated above.

[NOTE.—No other solution of this problem was received. Professor Zerr worked on this problem during the hot days of last August. He said the temperature was too high for him to make a complete solution. The problem is more difficult than we supposed. After trying to effect a solution the true character of the problem was revealed us. EDITOR.]

PROBLEMS.

26. Proposed by J. W. WATSON, Middle Creek, Ohio.

Find the average area of all right angled triangles having a given hypotenuse.

27. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Find the mean area of the *dodecagonal surface* formed by joining in order the points taken at random, one in each sector of a regular dodecagon.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

12. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

If the measures of curvature and tortuosity of a curve be constant at every point of a curve, the curve will be a helix traced on a cylinder.

Solution by the PROPOSER.

A helix, inclination ω, traced on a right circular cylinder, radius r, is an unicursal curve, and may be defined by the equations $x=r\cos\theta\ldots(1)$, $y=r\sin\theta\ldots(2)$, and $z=r\theta\tan\omega\ldots(3)$, in which θ is the angle through which the generating line has resolved when the point has moved through a space z on the generating line. From (1), (2), and (3), we have respectively

$$\frac{dx}{d\theta}=-r\sin\theta;\ d\left(\frac{dx}{d\theta}\right)=\frac{d^2x}{d\theta^2}=-r\cos\theta\ldots(4),$$

$$\frac{dy}{d\theta}=r\cos\theta;\ d\left(\frac{dy}{d\theta}\right)=\frac{d^2y}{d\theta^2}=-r\sin\theta\ldots(5),$$

$$\frac{dz}{d\theta}=r\tan\omega;\ d\left(\frac{dz}{d\theta}\right)=\frac{d^2z}{d\theta^2}=0\ldots(6).$$

$$\therefore\ \frac{ds}{d\theta}=\sqrt{\left[\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2+\left(\frac{dz}{d\theta}\right)^2\right]}=\frac{r}{\cos\omega}\ldots(7).$$

Dividing (4), (5), and (6), by the square of (7),

$$\frac{d^2x}{ds^2}=\frac{-\cos\theta\cos^2\omega}{r}\ldots(8),\ \frac{d^2y}{ds^2}=\frac{-\sin\theta\cos^2\omega}{r}\ldots(9),\ \text{and}\ \frac{d^2z}{ds^2}=0\ldots(10).$$

Since the reciprocal of the radius of curvature is the *measure of the curvature* at any point of a tortuous curve, we have

$$\frac{1}{\rho}=\sqrt{\left[\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2\right]}=\frac{\cos^2\omega}{r}\ldots\ldots(11),$$

which is necessarily a *constant* quantity for every point of the curve.

The formula for the *measure of the tortuosity* at any point of a tortuous curve is, regarding τ as the radius of torsion,

$$\frac{1}{\tau}=\sqrt{\left[\left(\frac{d\lambda}{ds}\right)^2+\left(\frac{d\mu}{ds}\right)^2+\left(\frac{d\nu}{ds}\right)^2\right]}\ldots\ldots(12),$$

in which $\frac{d\lambda}{ds}=\frac{d}{ds}\left[\rho\left(\frac{dy}{ds}\cdot\frac{d^2z}{ds^2}-\frac{dz}{ds}\cdot\frac{d^2y}{ds^2}\right)\right]\ldots\ldots(a),$

$$\frac{d\mu}{ds}=\frac{d}{ds}\left[\rho\left(\frac{dz}{ds}\cdot\frac{d^2x}{ds^2}-\frac{dx}{ds}\cdot\frac{d^2z}{ds^2}\right)\right]\ldots\ldots(b),$$

$$\frac{d\nu}{ds}=\frac{d}{ds}\left[\rho\left(\frac{dx}{ds}\cdot\frac{d^2y}{ds^2}-\frac{dy}{ds}\cdot\frac{d^2x}{ds^2}\right)\right]\ldots\ldots(c).$$

From (4), (5), and (6), by means of (7), we deduce

$$\frac{dx}{ds}=-\cos\omega\sin\theta\ldots\ldots(13,\ \frac{dy}{ds}=\cos\omega\cos\theta\ldots\ldots(14),\text{ and }\frac{dz}{ds}=\sin\omega\ \ldots\ldots(15).$$

Reducing (a), (b), and (c), by means of (7), (8), (9), (10), (13), (14), (15); and then differentiating the results, we have respectively

$$\frac{d\lambda}{ds}=\frac{\sin\omega\cos\omega\cos\theta}{r}\ldots\ldots(16),\quad \frac{d\mu}{ds}=\frac{\sin\omega\cos\omega\cos\theta}{r}\ldots\ldots(17),$$

and $\frac{d\nu}{ds}=\frac{d}{ds}\left[\frac{r\cos^3\omega(\sin^2\theta+\cos^2\theta)}{\cos^2\omega}\right]=0\ldots\ldots(18).$

Transforming (12) by means of (16), (17), and (18),

$$\frac{1}{\tau}=\sqrt{\left[\frac{\sin^2\omega\cos^2\omega(\sin^2\theta+\cos^2\theta)}{r^2}\right]}=\frac{\sin\omega\cos\omega}{r}\ldots\ldots(19),$$

which is also necessarily a *constant* quantity for every point of the curve.

NOTE.—Multiplying the numerator and the denominator of the right-hand member of (19), by $\cos\omega$, we have

$$\frac{1}{\tau}=\frac{\sin\omega}{\cos\omega}\times\frac{\cos\omega}{r}=\tan\omega\times\text{the curvature}.$$

If $\omega=\frac{1}{4}\pi$, the curvature and the tortuosity are necessarily equal. Had we assumed

$$x=\left(\frac{s}{\sqrt{6}}\right)\cos\left[\left(\frac{1}{\sqrt{2}}\right)\log\left(\frac{2s^2}{3c^2}\right)\right],\ y=\left(\frac{s}{\sqrt{6}}\right)\sin\left[\left(\frac{1}{\sqrt{2}}\right)\log\left(\frac{2s^2}{3c^2}\right)\right],$$

and $z=s\div\sqrt{2}$, then $1/\rho$ and $1/\tau$ would each have equaled $1/s$; that is, the curvature and the tortuosity would then have been the same for every point of the curve. Truly, the helix is a wonderful curve: it can easily be *axised* and *pitched* so as to have the same curvature and tortuosity as any given curve, while the loci of the centers of curvature and tortuosity are similar helices traceable on the same cylinder.

Also solved by *Professor G. B. M. Zerr.*

QUERIES AND INFORMATION.

RUSSIAN SCIENCE NOTES.

By Professor G. B. HALSTED, University of Texas, Austin, Texas.

The Jubilee-book issued by the University Kasan in commemoration of the Lobachevsky Centenary has already reached a very great circulation. His compatriots are pushing the non-Euclidean geometry.

N. P. Sokolov has just issued at Kiev (University Press) a pamplet of 32 pages (large 8vo.) entitled "The significance of the researches of N. I. Lobachevsky in geometry."

Volume IV of the second series of the Bulletin of the physico-mathematical society of Kasan, pp. 18-41, contains an interesting contribution by W. Sichstel on the fundamental theorems of spherical geometry.

Two books on America have lately been published in Russian. One is by Witkowsky, a scientist sent by the Russian government to study geodetic work in the United States. The other is published by a Russian, now resident in Los Angeles, who has been more than ten years in America, and has here amassed a fortune. He is a fervid republican, and writes under the *nom-de-plume* Tverski.

The well-known and justly admired writer Korolenko, ranked by the Russians second only to Tolstoi of living authors, was, during 1893, in America, and is about to issue his impressions of travel. This book, because of the high reputation of the author, is awaited with keen interest.

ARE LOBATSCHEWSKY'S PRINCIPLES APPLICABLE TO MECHANICS?

By WARREN HOLDEN, Professor of Mathematics, Pennsylvania Girard College, Philadelphia.

1. Is it expected that, when Lobatschewsky's new geometry is generally accepted, it will so permeate the old geometry as to modify its practical applications? Will the problems of engineering and construction have to be reconsidered?

Take a single example.

The builder of a railroad is very careful to keep the two rails of the track "everywhere equidistant." Unless they were parallel in the Euclidian sense disaster to a passing train would reveal the fact. Would Lobatschewsky's parallels, which always approach each other, answer for railroads?

2. Upon the surface of a sphere take a triangle with a given length of sides. The greater the diameter of the sphere the nearer the surface enclosed by the triangle approximates a plane, and simultaneously the nearer the sum of the angles of the triangle approaches two right angles. In passing from

surfaces, on which the angle sum is greater than two right angles, to surfaces, if any such there be, on which the angle sum is less than two right angles, we must come to a surface on which said sum is equal to two right angles. If this surface be not the Euclidian plane, what is it?

PROFESSOR SCHEFFER'S QUERY.

QUERY: Can any one furnish a reason for the peculiar name of the "devil's curve," or for the name which Professor Matz employs? J.F.W.S.

ANSWER.

"The Devil on Two Sticks" is best described as *a horizontal top.* According to the *Century Dictionary*, these toys are turned out of hard wood. Inclose the figure 8 with braces, { 8 }; take a strong cord about four feet in length, and fasten one end of this cord in the concave part of each cusp of the braces. These braces are *distorted* branches of "la curbe du diable;" and they represent the *sticks*, by means of which the cord is rapidly manipulated in spinning the devil. Loop the cord around the waist of the devil; then bring the two sticks into a vertical plane; and acquire skill in balancing the devil and getting him into a motion of rotation about a vertical axis, by means of the friction of the cord. Unless you *spin* the devil in a *lively* manner, he will fall off the chord and hurt your toes in revenge! As to the genesis and history of this curve and its branches, all appears to be obscure; Cramer may have given the name "la courbe du diable," since this curve clearly characterizes the *outline* and *symmetry* of the toy—"The Devil on Two Sticks." So far as is known to Professor William Woolsey Johnson and the writer, Cramer is the first to use the French name of this curve. Undoubtedly *Routh's Rigid Dynamics* and other standard works on *Rotation*—not omitting Professor Ziwet's excellent *Treatise on Theoretical Mechanics*, would furnish very desirable theoretical knowledge in mathematically spinning "The Devil on Two Sticks."—F. P. MATZ.

ARTHUR CAYLEY.

By Professor G. B. HALSTED, University of Texas, Austin, Texas

How Professor Cayley touched every thing mathematical, and touched nothing which he did not adorn, may be illustrated by the following unpublished letters, which were the first expression of discoveries that have since taken their permanent place in our best text-books.

They are both the outcome of the sudden and fruitful interest in *linkage*, dating from Sylvester's interview with Tchebychev, when, leaving behind him the diagram of the now celebrated Peaucellier's Cell, the illustrious Russian gave in parting the characteristic advice: "Take to Kinematics, it will repay you; it is more fecund than geometry; it adds a fourth dimension to space."

I will transcribe the letters exactly, not only because the recent death of Tchebychev, followed in less than two months by that of Cayley, gives them now a special pertinence, but because it is of interest to compare one with what is given on "tram motion" in Kempe's "How to Draw a Straight Line," and the other with its reproduction by no less a master than Clifford on pages 149, 150 of his Dynamic, whence I add figure 2.

"Roberts' theorem of 3 bar motion takes the following elegant form:

Take a triangle ABC & a point O and thro' O draw lines ‖ to the sides as in the figure—the 3 shaded △'s are of course similar to ABC.

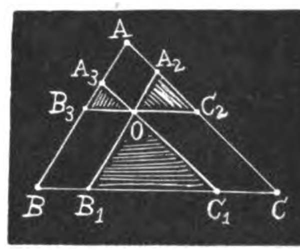


Fig. 1.

Now imagine a linkage composed of the shaded △'s and the bars AA_2, AA_3, BB_3, BB_1, CC_1, CC_2 pivotted together at A, B, C, A_2, A_3, B_3, B_1, C_2, C_1, O: then however the figure is moved, [of cource A_3, B_3 do not continue in the line AB, &c.], the triangle ABC *will remain similar to the shaded triangles*: and if in any position of the figure we fix the points A, B, C, then the point O will be moveable in a curve, viz. we have the *same* curve described by O considered as the vertex of OA_3B_3, where the two radii are AA_3, BB_3—by O considered as the vertex of OA_2C_2 &c.—and by O considered as the vertex of OB_1C_1, &c. Cambridge 22nd, Feby. 1876.

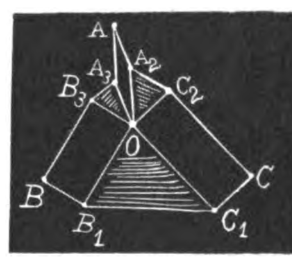


Fig. 2.

The porism is *very* pretty: it was new to me, tho' I think it ought not to have been so.

Look at the theorem thus. Imagine a plane, 2 points thereof A, C moving in fixed lines $O\alpha$, $O\gamma$.

Describe the circle OAC, which consider as a circle fixed in the plane & moveable with it. Then the theorem is that any point B of this circle moves in a line $O\beta$ thro' O.

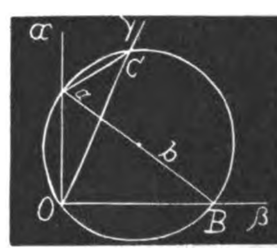


Fig. 3.

In particular B may be the opposite extremety of the diameter thro' A: and we have then the points A, B moving on the lines $O\alpha$, $O\beta$ at right angles to each other. Viz. the general case of a plane moving two points thereof on two fixed lines is reduced to this well-known particular case. And the theorem comes to this, that dividing the rod AB at pleasure into two parts AM, MB, and drawing MC at rt. angles and a mean proportional, the locus of C is a right line thro' O, which is of course easily proved."

Yours very sincerely, A. Cayley.

Cambridge, 5th May.

EDITORIALS.

THIS issue of the MONTHLY was mailed March 20th.

PERSONS failing to receive the MONTHLY shortly after the 25th of each month, should notify the PUBLISHERS at once.

OUR contributors will please send the material for the several departments to the editor of the respective departments and thus avoid delay and insure proper credit.

CARE should be taken in sending money in payment of subscriptions. The editors can not be responsible for money sent in any way other than by money order, draft, express money order, or registered letter. All money, drafts, etc., should be sent to B. F. FINKEL, Kidder, Missouri.

THROUGH an oversight, Dr. Matz was not credited with two different solutions of problem 11, Miscellaneous Department, nor with two different solutions of problem 26, Department of Calculus.

WE are pleased to note PROF. P. H. PHILBRICK is recovering from the effects of a severe attack of La Grippe.

DEPARTMENT F.—Mathematics of the University Extension Summer Meeting of the University of Pennyslvania will be under the direction of Dr. I. J. Schwatt of the University of Pennsylvania. The meeting will be held from July 1st to July 27th. Classes will be formed in all departments of mathematics. On Monday evening July 1st, Dr. Schwatt will deliver to the students of all departments of the summer meeting, an address on The Importance of a Mathematical Training to Students of Various Branches of Science.

THROUGH the kindness of Dr. Halsted we have received five addresses delivered before the Texas Academy of Science. These addresses are given by men who are eminent in their special departments of science, and are both interesting and instructive.

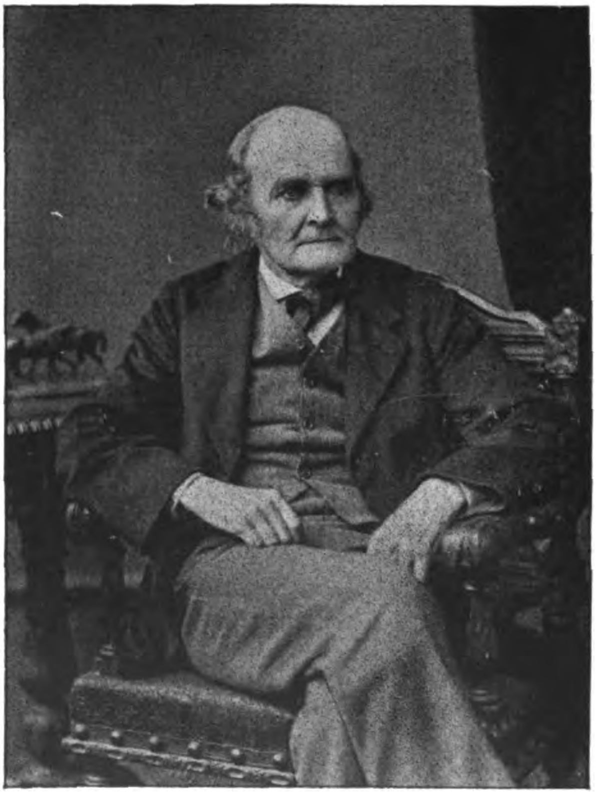
WASHINGTON, D. C., Feb. 18, 1895.

PROF. B. F. FINKEL, A. M.,

MY DEAR SIR: Enclosed you will find a money order of $3.00, which you will please accept as my subscription of the AMERICAN MATHEMATICAL MONTHLY for 1895. It is worth even more than that; and I fully agree with Prof. Matz, (a native of the same state and county with me), that it is almost impossible to give so much mathematics for $2.00.

With best wishes for its success, and hoping that by the end of 1895 it shall have been put on a paying basis, I remain, with pleasure, its regular devoted reader and occasional contributor, M. A. GRUBER.

We have taken the privilege of publishing Mr. Gruber's letter in full, as a specimen of some of the encouraging words that come to the MONTHLY from time to time.



ARTHUR CAYLEY.

THE AMERICAN MATHEMATICAL MONTHLY.

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I. BIOGRAPHY.

ARTHUR CAYLEY.

BY DR. ALEXANDER MACFARLANE.

ARTHUR CAYLEY was born at Richmond in Surrey, England, August the 16th, 1821. His father, Henry Cayley, was descended from the Cayleys of Brompton, in Yorkshire, but was at the time a merchant of St. Petersburg where he had married a Russian lady. In 1829 his parents took up their permanent residence at Blackheath in England; and Arthur was there educated at a private school for four years. At the age of 14 he was sent to King's College School, London; and the master of that school having observed the promise of a mathematical genius advised the father to educate his son not for his own business, but to enter the University of Cambridge.

In 1838 Arthur Cayley entered Trinity College, Cambridge, at the rather early age of 17. Throughout his undergraduate course he was first at his college examinations by an enormous interval, and he finished his undergraduate career in 1842 by carrying off the two highest honors, namely, the first place, or Senior Wrangler, in the Mathematical Tripos, and the first prize in the competition for the Smith Prizes. Immediately elected a Fellow of his College, he continued to reside at Cambridge for several years, during which time he lectured on mathematics, and also contributed papers to the *Cambridge Mathematical Journal*. His first contribution to that Journal was made, when he was an undergraduate, in 1841.

At that time it was necessary for a Fellow to take Holy Orders, or else resign the fellowship at the end of seven years. Mr. Cayley chose the latter alternative, and became by profession a conveyance in Lincoln's Inn, London. He followed that profession for 14 years with conspicuous ability and success, and at the same time made many of his most important contributions to mathematical science.

About 1861 the Lucasian professorship of mathematics at Cambridge — the chair made illustrious by Sir Isaac Newton—fell vacant; it was filled by G. G. Stokes, already eminent for his work in mathematical physics, and Senior Wrangler the year before Cayley. However, it was felt desirable to secure Cayley also, and for this purpose the Sadlerian professorship of mathematics was created, which resulted in Cayley marrying and settling down at Cambridge, in 1863.

The duties of the Sadlerian professor were defined as follows: "to explain and teach the principles of pure mathematics, and to apply himself to the advancement of the science". In carrying out the former part of the duties Professsor Cayley did not give the same course of lectures year after year, but each year took for his subject that of the memoir on which he was engaged. As a consequence his students were few, for advanced work of that kind did not pay in the great mathematical examination. How well he carried out the second part of the duties may be inferred from the fact that the Royal Society Catalogue of Scientific Papers enumerates 430 memoirs contributed by him between the years 1863 and 1883, making a total up to the latter date of 724. As he continued active to the last, it is probable that the grand total of his papers does not fall short of 1000. Some of his most celebrated contributions are: Chapters in the Analytical Geometry of (n) Dimensions, On the theory of Determinants, On the theory of linear transformations, Ten Memoirs on Quantics, Memoir on the theory of Matrices, Memoirs on Skew Surfaces, otherwise Scrolls, On the Motion of Rotation of a solid Body, On the triple tangent planes of surfaces of the third order. Several of his achievements are elegantly referred to in a poem written by his colleague Clerk Maxwell in 1874, and addressed to the Committee of subscribers who had charge of the Cayley Portrait Fund:

O wretched race of men, to space confined!
What honor can ye pay to him whose mind
To that which lies beyond hath penetrated?
The symbols he hath formed shall sound his praise,
And lead him on through unimagined ways
To conquests new, in worlds not yet created.

First, ye Determinants, in order row
And massive column ranged, before him go,
To form a phalanx for his safe protection,
Ye powers of the nth root of -1!
Around his head in endless cycles run,
As unembodied spirits of direction.

And you, ye undevelopable scrolls!
Above the host wave your emblazoned rolls,
Ruled for the record of his bright inventions.
Ye cubic surfaces! by threes and nines
Draw round his camp your seven and twenty lines
The seal of Solomon in three dimensions.

March on, symbolic host! with step sublime,
Up to the flaming bounds of Space and Time!
There pause, until by Dickenson depicted,
In two dimensions, we the form may trace
Of him whose soul, too large for vulgar space,
In *n* dimensions flourished unrestricted.

The portrait was presented to Trinity College, and now adorns their Hall. He is represented as seated at a desk, with quill in hand, and thinking out intently some mathematical idea.

But mathematical science was advanced by Professor Cayley in yet another way. By his immense learning, his impartial judgment, and his friendly sympathy with other workers, he was eminently qualified to act as a referee on mathematical papers contributed to the various societies. Of this kind of work he did a large amount, and of his kindliness to young investigators I can speak from personal experience. Several papers which I read before the Royal Society of Edinburgh were referred to him, and he recommended their publication. Some time after I attended a meeting of the Mathematical Society of London, but the friend who would have introduced me could not be present. Professor Cayley was present, and on finding out who I was, gave me a cordial handshake, and referred in the kindest terms to the papers he had read. His was a cosmopolitan spirit, delighting only in the truth, and friendly to all seekers after the truth.

Among Cayley's papers there are several on a "Question in the Theory of Probabilities". The question was propounded by Boole, and he applied to its solution the general method of "The Laws of Thought". It was afterwards discussed by Wilbraham, Cayley and others in the *Philosophical Magazine*. My attention was drawn to the question when writing the *Principles of the Algebra of Logic*, and I ventured to contribute my idea of the question to the *Educational Times*. On mentioning the matter to Professor Kelland, he intimated pretty plainly that the discussion had been closed by Professor Cayley, and that it was temerity on my part to write anything on the subject. But the great mathematician did not think so; he wrote me a letter discussing the question and my particular way of viewing it, as well as the fundamental ideas in which I differed from Boole.

In 1882 he received a flattering invitation from the trustees of the Johns Hopkins University to deliver a course of lectures on some subject in advanced mathematics. He chose as his subject the Elliptic and Abelian functions; and the impression which his presence created has been well described by Dr. Matz in his brief notice in the January number of the MONTHLY.

Next year he was president of the British Association at the Southport meeting. In his address he spoke of the foundations of mathematics, reviewed the more important theories, traced the connection of pure with applied mathematics, and gave an outline of the vast extent of Modern Methematics.

He regarded the complex number $a+bi$ as the fundamental quantity of mathematical analysis, and considered that with such a basis, algebra was a complete and bounded science, in which no further imaginary symbols could spring up. It is the more remarkable that he held such a view, when we consider that early in his career he made a notable contribution to space analysis. Starting from Rodrigues' formulae for the rotation of a solid body, he arrived at the quaternion formula, and was anticipated by Hamilton only by a few months. But Cayley took a Cartesian view of analysis to the last, as is evident from the chapter which he contributed to Tait's *Treatise on Quaternions*. His aim there is to give an analytical theory of quaternions. Hamilton's aim on the other hand was to give a quaternionic theory of analysis. The difference is brought out still more strikingly in a paper printed in the last number of the *Proceedings of the Royal Society of Edinburgh*.

In 1889 the Cambridge University Press commenced the re-publication of his mathematical papers in a collected form. It was calculated that they would occupy 10 quarto volumes; 7 volumes have already appeared; and it is believed that 12 volumes will be required. No mathematician has ever had his works printed in a more handsome manner. In addition he is the author of a separate work on *Elliptic Functions*.

Space fails to enumerate the honors which he received from Universities and Scientific Academies both of the Old and of the New World. But we may mention specially, that from the Royal Society he received a Royal Medal and a Copley Medal; from the Mathematical Society of London the first De-Morgan Medal; and at the instance of the President and Members of the French Academy he was made an Officer of the Legion of Honour.

On the 26th of January he died at Cambridge. His body was laid to rest in Mill Road Cemetery in the presence of official representatives from foreign countries and many of the most illustrious philosophers of England. His spirit still speaks to us from his works, and will continue to speak to many succeeding generations.

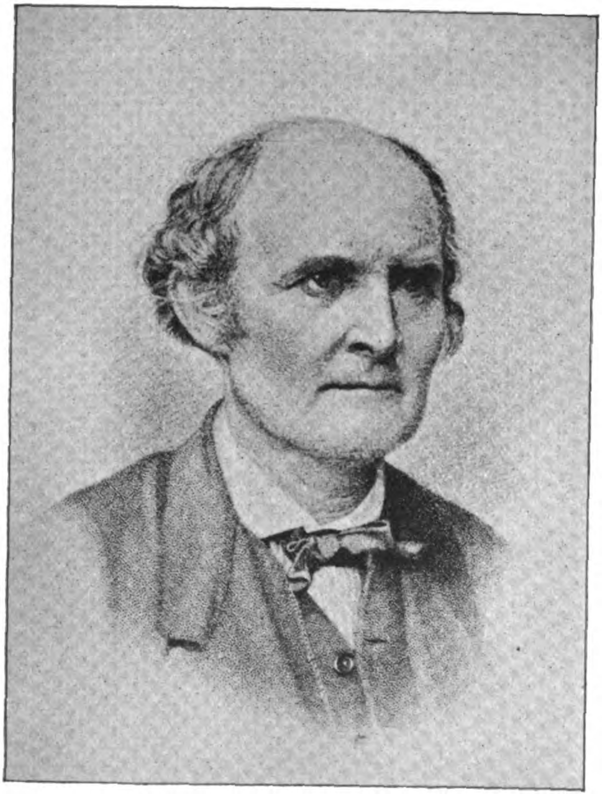
II. BIOGRAPHY.

ARTHUR CAYLEY.

BY DR. GEORGE BRUCE HALSTED.

On January 26th, 1895, after long suffering, passed away Professor Cayley, one of the very greatest masters ever known to the world of pure mathematics.

Of the great quaternion of Senior Wranglers of 1840 Leslie Ellis, 1841 Stokes, 1842 Cayley, 1843 Couch Adams, the second alone now remains.



ARTHUR CAYLEY.

Arthur Cayley was born on August 16th, 1821. His ancestor Roger de Cailli was a great lord in the reign of Henry II. His grandfather and father were both merchants in St. Petersburg. His father married a Russian, and though born in England, Cayley's mother tongue was Russian, and his features had a Russian cast. Like so many Russians, he spoke most European langua.es well.

Can there be anything in what has so often been cited as fact, that in the Russian race alone the brain of the woman equals that of the man in size and weight?

Arthur Cayley was a pupil of King's College School, London, and entered Trinity College, Cambridge, already a well-equipped mathematician, at the age of 17. In 1842 he took the two highest honors in the University of Cambridge, he was Senior Wrangler and First Smith's Prizeman. At that time, more than half a century ago, the Senior Wrangler was almost always as a matter of course a Johnian, so a Trinity Senior Wrangler was apt to be an object of curiosity. One of his college mates describes him at that date as a crooked little man, in no respect a beauty, and not in the least a beau. On the day of his triumph, when he was to receive his hard-earned honors in the Senate House, some of his friends combined their energies to dress him, and put him to rights properly, so that his appearance might not be altogether unworthy of his exploits and his College. He was already a man of much varied information, and that on some subjects the very opposite of scientific; for instance he was well up in all the current novels, an uncommon thing at Cambridge, where novel-reading then was not one of the popular weaknesses. His Johnian competitor for first place was a fearfully hard student, and had once worked *twenty hours a day* for a week together at a College examination. But now he almost broke down from over exertion just as the time of trial was coming on, and actually carried a supply of ether and other stimulants into the examination, in case of accidents. Nevertheless he made a good fight of it, and having great *pace* as well as *style* in addition to his knowledge, beat Cayley a little on the bookwork, but was beaten two hundred marks in problems, which decided the contest.

One of the low bookwork papers to which three hours were allotted happening to be rather shorter than usual, the man from St. John's, either as a bit of bravado to frighten his opponent, or because having done all that could be done he had no reason for waiting longer, came out at the expiration of two hours, having floored the paper in that time. His early exit did not escape notice, and the same evening a Trinity Senior Soph rushed up in great fear to the room of his friend, on whom the hopes of the College depended. "Cayley! Cayley! they tell me S— floored the paper this afternoon in two hours. Is it so?" The mathematician, who was refreshing himself after the fatigues of the day with the innocent and economical luxury of a footbath, looked up at the querist from his tub with the equanimity of a Diogenes, and replied: "Likely enough he did. I floored it myself in two hours and a half". The examination for the Smith's Prizes which took place immediatly after the result of the Mathematical Tripos was declared, had a similar result; Cayley beat his opponent, but with nothing to spare. The matter was very different the next year, when Couch Adams, the discoverer of Neptune, won not only easily, but had three thousand marks to the Second Wrangler's fourteen hundred, so that there was more numerical difference between them than between the Second Wrangler and the *spoon*, or last man. But this was produced by a singular case of fright or stampede which occurred at this examination. The man who would have been second, like Adams a Johnian, took fright when four of the six days were over, and actually ran away, not only from the examination but out of Cambridge, and was not discovered by his friends or family till some time after. Even as it was, and without the last two days, he came out ninth in the list of wranglers. But even if Cayley had been beaten for first place, he might still have been equally as eminent as now; for has not Cambridge that other tremendous tetrad, Sylvester, Wm. Thomson, Clerk-Maxwell, Clifford, all Second Wranglers!

In 1841 Cayley published his first paper, thus commencing the astounding series of over 800 memoirs with which he so enriched his science. The collected edition of his works now being published by the University Press will extend to ten or more quarto volumes, a scientific monument equally unique in amount, range, and quality.

After his election to a Fellowship, which, as he was unwilling to take Holy Orders, could be only temporary, he studied conveyancing in London, and at Lincoln's Inn first met his greatest and lifelong friend and fellow-genius Sylvester, for they had never met at Cambridge, where Sylvester was Second Wrangler in 1837.

He practised as a conveyancer for 14 years, but during this time his real occupation was pure mathematics, and in those years some of his most notable discoveries were made. The law was always drudgery to him. The superabundant verbiage of legal forms was always distasteful to him. He once remarked that "the object of law was to say a thing in the greatest number of words, of mathematics to say it in the fewest."

Cayley was a very gentle, sweet character. Sylvester told me that he never saw him angry but once, and that was when a messenger broke in on one of their interviews with a mass of legal documents, new business for Cayley. In an access of disgust, Cayley dashed the documents upon the floor.

In 1863 Lady Sadler's various trusts were consolidated, and a new Sadlerian Professorship of Pure Mathematics was created in the University of Cambridge, especially for Cayley. As chairman of the Association for Promoting the Higher Education of Women he did most to raise Newnham College to its present influential position.

In Cambridge he was accustomed to give the small classes of advanced students who were prepared to follow him no mere routine course, but, like the best German professors since Jacobi, the latest and highest work on which he was at the time engaged.

As early as 1852 he was a fellow of the Royal Society. In 1858 he joined Sylvester and Stokes in starting the Quarterly Journal of Pure and Applied Mathematics. In 1882 he delivered a special course of lectures at the Johns Hopkins University, where Sylvester was still professor. Baltimore was then the apex not only of the Western Continent, but of the world, for Salmon soon after said that if European mathematicians had to elect themselves a head, it would be Cayley. In 1863 he married and settled permanently in Cambridge.

Cayley was assuredly the most learned and erudite of mathematicians. Of him it might be said, he knew everything, and he was the very last man who ever will know everything. I have heard Sylvester say that when he wished to know anything he simply asked Cayley, for to Sylvester it was not only often irksome to study what had been done by others, but impossible, since the very beginning of such study was sure to start in him a train of original thought and research which absorbed him irresistably. This wideness of knowledge made Cayley invaluable as a mathematical referee. To the Royal Society, the Mathematical Society, the Royal Astronomical Society, the Cambridge Philosophical Society he was long the principal adviser as to the merits of mathematical papers presented for publication. Cayley's erudition gave his originality always the most fertile fields.

In 1841 the wonderful George Boole, the creator of algorithmic logic, made use of a simple case of what we would now call *invariance* in linear substitutions. Then Cayley set himself the problem to determine *a priori* what functions of the coefficients of a given equation possess this property. He called such functions hyperdeterminants, until Sylvester the mathematical Adam, who names the creatures, called them *invariants*. Substitutions and invariance are now the heart of the very latest analytic mathematics, and have received an extraordinary transformation and development at the hands of Sophus Lie.

Again, the idea that any metrical property in geometry could be looked upon as a projective relation in a particular configuration began to occur in the French school. For example Laguerre in 1853 so expresses an angle. But in 1859 in his sixth memoir on Quantics Cayley published his solution of the general problem he had set himself of finding a general theory of projective metrics of which ordinary metrics should be a special case; thus breaking down the distinction between pure positional or descriptive geometry and the ordinary metrical geometry by merging all into projective geometry.

Remembering that von Staudt had founded cross-ratio on a pure projective basis in his theory of the *Wurf*, entirely without using measurement in the ordinary sense (direct comparison as to size by congruence), Klein saw that Cayley's theory of projective measurement leads directly to the three possible cases of geometry, Euclidean and non-Euclidean, which he called parabolic, elliptic, hyperbolic. The hyperbolic is the now well-known non-Euclidean geometry of Lobachevsky and Bolyai. Thus Cayley's doctrine of "the absolute", already greatly admired, was given additional importance, and its creation will ever rank as one of the very greatest of his achievements.

As a third epoch-making production of his fertile and tireless genius we may mention the theory of matrices, on which multiple algebra is based. In this, as in the theory of invariants, Sylvester was his most brilliant co-adjutor. Cayley was a devoted admirer of Euclid. In his great address as President of the British Association, speaking of Greek mathematics he says: "But the earliest extant writings are those of Euclid (B. C. 285).

There is hardly anything in mathematics more beautiful than his wondrous fifth book: and he has also in the seventh, eighth, ninth, and tenth books fully and ably developed the first principles of the theory of numbers, including the theory of incommensurables". In the same address he says: "It is well known that Euclid's twelfth axiom, even in Playfair's form of it, has been considered as needing demonstration; and that Lobachevsky constructed a perfectly consistent theory, wherein this axiom was assumed not to hold good, or say a system of non-Euclidean plane geometry. There is a like system of non-Euclidean solid geometry. Riemann's view was that having *in intellectu* a more general notion of space (in fact a notion of non-Euclidean space), we learn by experience that space (the physical space of our experience), is, if not exactly, at least approximately, Euclidean space. But suppose the physical space of our experience to be thus only approximately Euclidean space, what is the consequence which follows? Not that the propositions of geometry are only approximately true, but that they remain absolutely true in regard to that Euclidean space which has been so long regarded as being the physical space of our experience.

The three geometries (spherical, Euclidean, and Lobachevsky's) should be regarded as members of a system—viz., they are the geometries of a plane (two-dimensional) space of constant positive curvature, zero curvature, and constant negative curvature respectively; or again they are the plane geometries corresponding to three different notions of distance; in this point of view they are Klein's elliptic, parabolic, and hyperbolic geometries respectively."

But here this imperfect sketch must stop. Enough that his life furthered in the highest degree the aim of his university, in the words of his mother's compatriot Lobachevsky, "not only to enlighten the spirit with knowledge, but also to inculcate virtues, to implant a desire for glory, a feeling of nobility, justice, and honor, of strict and sacred honesty, that would resist all cases of temptation, apart from any fear of punishment."

SOME NOTES ON THE THEORY OF PROBABILITY.

By Professor G. B. M. ZERR. A. M., Principal of High School, Staunton, Virginia.

Of the three memoirs written by Prevost and Lhuilier, the first entitled, "Sur les Probabilites," was read November 12th, 1795 and occupies pages 117-142 of the mathematical portion of the volume, "Memoires de l'Acad. .... Berlin. This important Memoir deals with the following venerable problem:

An urn contains m balls some of which are white and the rest black, but the number of each is unknown. Suppose p white balls and q black balls have been drawn and not replaced; required the probability that out of the next $r+s$ drawings r shall give white balls and s black balls.

This problem suggests the discussion of the more general problem: From an unknown number of balls each equally likely to be any of n colors $a_1+a_2+a_3+\ldots+a_n$ balls are drawn and turn out a_1 of the first color, a_2 of the second, a_3 of the third, ...a_n of the nth. If $b_1+b_2+b_3+\ldots+b_n$ more balls are drawn find the probability that b_1 are of the first color, b_2 of the second, ..., b_n of the nth. The problem will not be altered, if we suppose the balls arranges along a straight line of length unity on n differant portions of the line. Call the first portion x_{n-1}, the sum of the first and second, x_{n-2}, the sum of the first three, x_{n-3}, ... the sum of the first $(n-1)$, x_1, then we get for the required chance the following definite integral:

$$p=\frac{\underline{|b_1+b_2+b_3+\ldots+b_n}}{\underline{|b_1}\,\underline{|b_2}\,\underline{|b_3}\ldots\underline{|b_n}}\cdot\frac{\int_0^1\int_0^{x_1}\int_0^{x_2}\ldots\int_0^{x_{n-2}}(1-x_1)^{a_1+b_1}(x_1-x_2)^{a_2+b_2}(x_2-x_3)^{a_3+b_3}\ldots x_{n-1}^{a_n+b_n}dx_1dx_2dx_3\ldots dx_{n-1}}{\int_0^1\int_0^{x_1}\int_0^{x_2}\ldots\int_0^{x_{n-2}}(1-x_1)^{a_1}(x_1-x_2)^{a_2}(x_2-x_3)^{a_3}\ldots x_{n-1}^{a_n}dx_1dx_2dx_3\ldots dx_{n-1}}$$

$$=\frac{\underline{|b_1+b_2+b_3+\ldots+b_n}\;\underline{|a_1+b_1}\;\underline{|a_2+b_2}\;\underline{|a_3+b_3}}{\underline{|b_1}\;\underline{|b_2}\;\underline{|b_3}\ldots\underline{|b_n}\;\underline{|a_1}\;\underline{|a_2}\;\underline{|a_3}\ldots\underline{|a_n}}\cdot\frac{\ldots\underline{|a_n+b_n}\;\underline{|a_1+a_2+a_3+\ldots+a_n+n-1}}{\underline{|a_1+a_2+a_3+\ldots+a_n+b_1+b_2+b_3+\ldots+b_n+n-1}}.$$

When $n=2$,

$$p=\frac{\underline{|b_1+b_2}\;\underline{|a_1+b_1}\;\underline{|a_2+b_2}\;\underline{|a_1+a_2+1}}{\underline{|b_1}\;\underline{|b_2}\;\underline{|a_1}\;\underline{|a_2}\;\underline{|a_1+a_2+b_1+b_2+1}},$$ same result as given in the memoir above referred to.

When $n=3$,

$$p=\frac{|\underline{b_1+b_2+b_3}\ |\underline{a_1+b_1}\ |\underline{a_2+b_2}\ |\underline{a_3+b_3}\ |\underline{a_1+a_2+a_3+2}}{|\underline{b_1}\ |\underline{b_2}\ |\underline{b_3}\ |\underline{a_1}\ |\underline{a_2}\ |\underline{a_3}\ |\underline{a_1+a_2+a_3+b_1+b_2+b_3+2}}.$$

Let $\dfrac{|\underline{a_1+a_2+a_3+2}}{|\underline{a_1+a_2+a_3+b_1+b_2+b_3+2}}=A$, then the chance that all are of the first color is $p_1=\dfrac{|\underline{a_1+b_1+b_2+b_3}}{|\underline{a_1}}A$. $p_2=\dfrac{|\underline{a_2+b_1+b_2+b_3}}{|\underline{a_2}}A$ that all are of the second color. $p_3=\dfrac{|\underline{a_3+b_1+b_2+b_3}}{|\underline{a_3}}A$. that all are of the third color.

$$p_4=\frac{|\underline{b_1+b_2+b_3}\ |\underline{a_1+b_1+b_2}\ |\underline{a_3+b_3}}{|\underline{b_1+b_2}\ |\underline{b_3}\ |\underline{a_1}\ |\underline{a_3}}A,$$ that b_1+b_2 are of the first color and b_3 of the third.

$$p_5=\frac{|\underline{b_1+b_2+b_3}\ |\underline{a_1+b_3}\ |\underline{a_3+b_1+b_2}}{|\underline{b_1+b_2}\ |\underline{b_3}\ |\underline{a_1}\ |\underline{a_3}}A,$$ that b_3 are of the first color and b_1+b_2 of the third and so on for any combination.

When $a_1=5$, $a_2=3$, $a_3=2$, $b_1=b_2=b_3=1$.

Then $p=\frac{72}{455}$, $p_1=\frac{8}{65}$, $p_2=\frac{4}{91}$, $p_3=\frac{2}{91}$, $p_4=\frac{9}{65}$, $p_5=\frac{36}{455}$. The chance that there are none of the second color is $p_6=p_1+p_3+p_4+p_5=\frac{33}{91}$.

These numerical results are the same as those obtained by an algebraic solution of the same problem given by the late Professor Wolstenholme.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED. A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the March Number.]

PROPOSITION XVI. *By any quadrilateral ABCD, of which the four angles together are equal to, or greater, or less than four right angles, is established respectively the hypothesis of right angle, or obtuse angle, or acute angle.*

Proof. Join AC. The three angles of the triangle ABC (fig. 14.) will not be together equal to, or greater, or less than two right angles, without the three angles of the triangle ADC being themselves also together respectively equal to, or greater, or less than two right angles, lest obviously (by the preceding) from one of those triangles be established one hypothesis, and another from the other, against the fifth, sixth, and seventh propositions of this work.

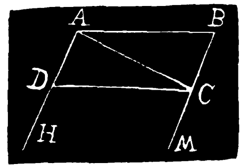


Fig. 14.

This holding good: If the four angles together of the premised quadrilateral are equal to four right angles, it follows that the three angles together of either of the just mentioned triangles will be equal to two right angles, and therefore (from the preceding) the hypothesis of right angle will be established.

But if indeed the four angles of this quadrilateral be together greater, or less than four right angles, similarly the three angles together of those triangles should be respectively either at the same time greater, or at the same time less than two right angles. Wherefore from these triangles would be established respectively (from the preceding) either the hypothesis of obtuse angle, or the hypothesis of acute angle.

Therefore by any quadrilateral, of which the four angles together are equal to, or greater, or less than four right angles, is established respectively the hypothesis of right angle, or obtuse angle, or acute angle. Quod erat demonstrandum.

Corollary. Hence: any two opposite sides of the premised quadrilateral being produced toward the same parts, as suppose AD to H, and BC to M; the two external angles HDC, MCD will be (Eu. I. 13.) either equal to, or less, or greater than the two internal and opposite angles together at the points A, and B, according as is true the hypothesis of right angle, or obtuse angle, or acute angle.

[To be continued.]

SOME SUGGESTIONS ABOUT VARIATION.

By ERIC DOOLITTLE, Professor of Mathematics, State University of Iowa, Iowa City, Iowa.

I think that in all of our Algebras the fundamental definitions of variation being in the idea of the ratio of two values of two changing quantities; the questions are first stated in the form of Proportions, and finally reduced to simple equations for solution.

It seems to me that this method is more cumbersome than necessary, especially in regard to the fundamental definitions, which thus involve such complex ideas that the student has a good deal of difficulty to grasp and apply them; at least that has been my experience with young students. I have therefore given the definitions in the form of equations in the first place, which seems to considerably simplify the theorems and their application.

Assuming the student to know what variables and constants are, the presentation of the subject might be somewhat as follows.

(1). If $x, y, z, w \dots$ are varying quantities, then v is said to be a *function* of $x, y, z, w \dots$ when any change in the value of any or all of these variables produces a change in the value of v. Thus v will vary when $x, y, z, w \dots$ vary.

(2). y is said to vary *as* x, if y always equals m times x, where m is constant, whatever be the value of x. This is the simplest kind of variation,* and is sometimes expressed by saying "y varies directly as x." Thus, if a train go m miles an hour, the distance (y) varies directly as the number of hours (x), since $y = mx$.

(3). y is said to vary *inversely* as x, if y always equals m times the inverse of x; that is, if $y = m(1 / x)$. Thus in the last illustration, if it require x hours for a train to go m miles, then the speed in miles per hour (y) varies inversely as x, since $y = \frac{m}{x}$.

(4). y is said to vary directly as $x, u, z, w \dots$ if y always equals m times the product $xuzw \dots$ where, as before, m is constant.

(5). Finally, y may be said to vary directly as certain quantities, and inversely as certain others, if y always equals m times the continued product of the former and the inverse of each of the latter. Thus y varies directly as z^2 and $a+x$, and inversely as $\sqrt[3]{x}$ if $y = mxz^2 \times (a+x) \times \frac{1}{\sqrt[3]{x}}$.

(6). The equations arising from the last four definitions may be called the *Statement* of the variation, and the first step toward the solution of any problem in variation is to write this statement. We then substitute in it such values as are known, and solve for what is required. If there are several different conditions in the problem, we make the statement for each separately, and solve the resulting simultaneous equations.

Numerous examples and illustrations of these principles should of course be given, and the proof of the more elementary theorems should follow. It will be seen that they are almost self evident by this method of treating the subject.

For instance, "If y varies as x, and x varies as z, then y varies as z":

"If y varies as x, and y' varies as x', then yy' varies as xx'":

"If y varies as zx, then x varies as y / z, and z varies as y / x" etc.

"If y is a function of two variables only, x and z; and if y varies as x when z is constant but when x is constant y varies as z, then y varies with x and z at once; that is $y = mxz$."

If the above definitions be admitted, the following proof of the last

* The distinction between variation in general and the simplest possible kind of it, is here introduced to guard against the supposition that all variation is of this simplest possible kind. This danger is pointed out by Dr. Chrystal on page 275 part 1 of his Algebra.

theorem may take the place of the longer one usually given:

"Since y varies with x, multiplying x alone will multiply y by the same factor; and similarly, multiplying z alone will multiply y; hence multiplying both x and z will twice multiply y, once by each of the respective factors. Hence x and z must enter as factors of the value of y, and since there are no other variable factors, y equals a constant expression times xz. That is $y=mxz$."

I suppose this subject of variation is pretty generally omitted by Preparatory and High School classes in Algebra. It seems to me to furnish an excellent opportunity to emphasize the difference between constant and variable quantities; a distinction the student usually meets here for the first time. I think it repays a few days careful work, by the introduction it thus gives to Analytic Geometry and the Calculus. Besides, by a few obvious applications to Astronomy and Physics, it can be made of interest to the pupil.

Such an oasis, after travelling in the desert of Radicals and Imaginaries, is very welcome.

TRUE PROPOSITIONS NOT INVALIDATED BY DEFECTIVE PROOFS.

By Professor John N. Lyle, Ph. D., Westminster College, Fulton, Missouri.

A bad cause may be brilliantly advocated and a good one poorly defended. A false proposition may be supported by plausible arguments and a true one by defective and even erroneous proofs. The true proposition is not thereby shown to be false or unworthy of acceptance.

John Playfair's demonstration of the angle-sum of a rectilineal triangle may be unsatisfactory and yet the proposition that the angle-sum is two right angles may be rigorously true and its contradictory absolutely false.

Legendre's demonstration that the angle-sum can not be less than two right angles is said by Professor Halsted to be "disgraceful." Even if this be admitted, it does not follow that the proposition itself should be doubted or rejected.

Discrediting Legendre's demonstration furnishes no legitimate warrant for postulating the truth of the hypothesis that the angle-sum can be less than two right angles.

The proofs that the angle-sum can be neither greater nor less than two right angles given in the pamphlet—Euclid and the Anti-Euclidians—may fall below the standard required by rigid geometrical science, but this does not justify the acceptance as true of the assumption that the angle-sum is greater or less than two right angles.

Lobatschewsky's theorem that the angle sum can not be greater than two right angles is manifestly in conflict with the doctrine of those metageometers who maintain that *the space in which we dwell* has constant, positive curvature and that the angle-sum of the rectilineal triangle drawn therein is greater than two right angles.

If it is maintained that the conclusions of Lobatschewsky, Riemann and Euclid are consistent with their respective premises, the question arises which of these systems is true. If any one does not really know which is right, confession of one's ignorance may be good for the soul, but can hardly be received as satisfactory evidence that the agnostic is in possession of geometrical science.

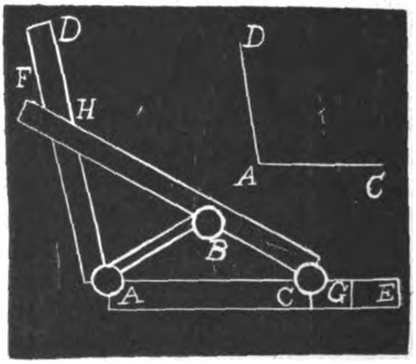
The hypothesis that Lobatschewsky, Euclid and Riemann all three tell the truth is confronted with the difficulty that they contradict each other.

Professor Halsted teaches as sound geometry the views of each of these three writers. I can not accept this teaching. If the Euclidian doctrine is true, according to logical law that which contradicts it must be false. This procedure of Professor Halsted antagonized the logical laws of non-contradiction and excluded Middle whether he is aware of it or not.

A TRISECTOR OF ANGLES.

By M. A. GRUBER, A. M., War Department, Washington, D. C.

Description. A, B, and C are centers and joints. G is a slide moving along the rule AE. The joint C is fixed to the slide so that the center C moves in the line AC. FC is a rule finely and accurately graduated from B to F, and fixed to the slide G by the joint C. AD is a fine and accurately graduated rule fixed to the rule AE by the joint A. AB is a small rule jointed at A and B.



Line AB equals line BC, both remaining constant.

The *edges* of the rules for use are those radiating from the centers.

Use. It is desired to trisect the $\angle DAC$.

Place the center A of the trisector upon the vertex A of the angle, so that the edge AC of the rule AE coincides with the side AC of the angle. Then move the rule AD until the edge coincides with the side AD of the angle. Now move the slide G until BH on the rule FC equals AH on the rule AD. Then draw a line along edge of rule AB.

$\angle BAC = \frac{1}{3} \angle DAC$. Bisect $\angle DAB$ and the trisection is complete.

Proof. $BC=AB$ and $BH=AH$ by construction.
$\angle HBA=\angle BAC+\angle ACA=2\angle BAC$. But $\angle HBA=\angle HAB$.
$\therefore\ \angle HAC=\angle HAB+\angle BAC=3\angle BAC$.

Within reasonable limits of length of the rules FC and AD, angles up to 120° can be trisected.

History. Last February four years ago, I was experimenting with triangles. I had drawn a rt.△ whose acute angles were 60° and 30°. By joining the vertex of the rt.∠ with the middle of the hypothenuse, I noticed that the rt.∠ was trisected. To devise an instrument for the trisection of any angle then engaged my mind for a few weeks, and the above device was the result.

I communicated my discovery to several mathematicians and inquired as to its practicability. The replies were not encouraging. One reason given was that an instrument with several joints and a slide, was not sufficiently accurate. The suggestion was also made that it would not pay to get it patented, as the trisecting of angles entered to a very limited extent in the mechanical applications.

Thinking that the readers of the AMERICAN MATHEMATICAL MONTHLY might be interested in this device, though it may be but a mathematical curiosity, I have given the foregoing brief sketch of it.

DIAGRAM FOR THE LAWS OF THE FALLING BODIES.

By Rev. A. L. GRIDLEY, Pastor of the Congregational Church, Kidder, Missouri.

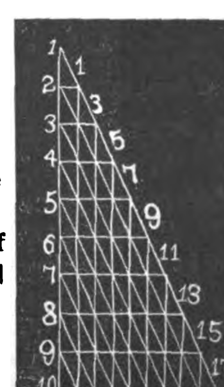
Let the distance a body would fall in one second be represented by one of the small triangles in diagram as a. During the first second it would fall through the firstspace, or triangle at the apex. During the second second it would pass through three, as that is the number of triangles in the second space which is indicated by the figures at the right. During the two second it would pass through $3+1$ triangles$=4a$, or $2^2\times a$.

To illustrate farther. How far would a body fall during the 9thsecond of its descent?

Opposite the figure 9 on the left are 17 triangles so it would pass through 17 times the distance it did during the first second or $17a$. How far would it fall during the ninth second without increment?

Leave off the right hand triangle and there would remain 18 so it would fall $18a$.

What would be the velocity at, say, the end of the 8th second? It would be the distance it would fall during the 9th second without increment, or the triangle at right hand side, $=16a$.



How far would it fall in, say, 9 seconds. Of course $9^2\,a$ or the sum of all the triangles in the first nine spaces.

With what velocity must a body be projected upward in order to rise during 10 seconds? Opposite 10 are 19 triangels so the initial velocity should be $19a$.

By a little thought any rule or problem in falling bodies can be *counted* out upon the diagram and it is unnecessary to commit any rule to memory as it can be produced at any moment from the diagram. Even the recollection will usually be sufficient to solve an ordinary problem as it has done with the inventor of the diagram—the writer—for thirty-five or forty years.

ERRATUM.—Owing to the extravagance of the compositor a needless *the* was inserted in the title of this paper.—PUBLISHERS.

ARITHMETIC.

Conducted by B.F.FINKEL, Kidder, Missouri. All Contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

38. Proposed by J. A. CALDERHEAD, B. Sc., Superintendent of Schools, Limaville, Ohio.

What must be the thickness of a 36-inch shell, in order that it may weigh 1 ton, supposing a 13-inch shell to weigh 200 pounds, when two inches thick.

IV. Solution by the PROPOSER.

$$200:2000::13^3-9^3:36^3-r^3;\text{ whence } r=31.74 \text{ inches.}$$

$$\therefore\ (36-31.74)\div2=2.13 \text{ inches}=\text{thickness of 36-inch shell.}$$

39. Proposed by P. C. CULLEN, Superintendent of Schools, Brady, Nebraska.

A, B, and C start from same point at same time. A north at rate of three miles per hour, B east at rate of four miles and C west at rate of five miles per hour. B at end of two hours starts at such an angle as to intersect A. How long after starting must C start north-west in order to meet A and B at common point?

II. Solution by Professor H. W. DRAUGHON, Olio, Mississippi.

While B travels 8 miles east, A travels 6 miles north. The rest of A's distance north, and the distance B travels after turning, are in the ratio of 3 to 4. Since B's latter distance is on the hypothenuse of a right triangle, whose base is 8 miles and perpendicular, A's distance, we have from Geometry, (hypothenuse+8)(hypothenuse−8)=($\frac{3}{4}$hypothenuse+6)^2 = $\frac{9}{16}$(hypothenuse+8)^2: whence, by division, we get hypothenuse−8=$\frac{9}{16}$(hypothenuse+8). $\therefore$ hypothenuse=28$\frac{4}{7}$ miles; and the perpendicular, = A's distance north.

$=\sqrt{(28\frac{4}{7})^2-8^2}=27\frac{3}{7}$ miles. Now C's route forms with A's route a right triangle whose perpendicular is $27\frac{3}{7}$. The sum of the hypothenuse and base $=C$'s distance $=\frac{5}{3}$ of A's distance $=27\frac{3}{7}\times\frac{5}{3}=45\frac{5}{7}$ miles. Also, from Geometry, the difference between the hypothenuse and base $=(27\frac{3}{7})^2\div45\frac{5}{7}=16\frac{16}{35}$ miles.

$\therefore$ Base $=\frac{1}{2}(45\frac{5}{7}-16\frac{16}{35})=14\frac{22}{35}$ miles. C's time in base is therefore, $14\frac{22}{35}\div5=2\frac{162}{175}$ hours $=2$ hours 55 minutes $32\frac{4}{7}$ seconds.

42. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

If $m=2$ct. be the interest on $M=100$ct. for $p=40$ days, find the yearly rate per cent.

I. **Solution by P. S. BERG, Apple Creek, Ohio.**

$\frac{1}{9}$ cent = the interest on 100 cents for 40 days at 1%.

$2\div\frac{1}{9}=18$. Hence, 18 is the yearly rate per cent.

II. **Solution by COOPER D. SCHMITT, Professor of Mathematics, Vanderbilt University, Knoxville, Tennessee.**

If m is the interest on M cents for p days, then $\frac{m}{p}$ is the interest on M cents for 1 day, and $\frac{360m}{p}$ is the interest on M cents for 360 days.

Hence the per cent will be $\frac{360m}{p}$ of $100=\frac{36000m}{Mp}$ %. If $m=2, p=40$, and $M=100$, the rate is $\frac{72000}{100\times40}=18$%.

Solutions of this problem were received from Professors *Matz* and *Zerr*.

43. **Proposed by B. F. BURLESON, Oneida Castle, New York.**

A, in a scuffle, seized on $\frac{2}{3}$ of a parcel of sugar plums; B caught $\frac{3}{8}$ of it out of his hands, and C laid hold on $\frac{3}{10}$ more: D ran off with all A had left, except $\frac{1}{7}$ which E afterwards secured slyly for himself; then A and C jointly set upon B, who, in the conflict, let fall $\frac{1}{2}$ he had, which were equally picked up by D and E, who lay perdu. B then kicked down C's hat, and to work they all went anew, for what it contained; of which, A got $\frac{1}{4}$, B $\frac{1}{3}$, and D $\frac{2}{7}$, and C and E equal shares of what was left of that stock. D then stuck $\frac{3}{4}$ of what A and B last acquired, out of their hands; they, with difficulty, recovered $\frac{5}{8}$ of it in equal shares again, but the other three carried off $\frac{1}{8}$ apiece of the same. Upon this, they called a truce, and agreed that the $\frac{1}{7}$ of the whole, left by A at first, should be equally divided among them. How much of the prize, after this distribution, remained with each of the competitors?

I **Solution by A. L. FOOTE, C. E., Middleburg, Connecticut.**

First, A has $\frac{2}{3}$; second, A has $\frac{2}{3}-(\frac{3}{8}+\frac{3}{10})$ of $\frac{2}{3}=\frac{13}{60}$. B has $\frac{3}{8}$ of $\frac{2}{3}=\frac{1}{4}$, and C has $\frac{3}{10}$ of $\frac{2}{3}=\frac{1}{5}$; third, A has $\frac{13}{60}-(\frac{13}{70}+\frac{13}{420})=0$, B has $\frac{1}{4}$, C, $\frac{1}{5}$, D $\frac{6}{7}$ of $\frac{13}{60}=\frac{13}{70}$, and E, $\frac{1}{7}$ of $\frac{13}{60}=\frac{13}{420}$; fourth, A has 0, B has $\frac{1}{2}$ of $\frac{1}{4}=\frac{1}{8}$. C has $\frac{1}{5}$, D has $\frac{13}{70}+\frac{1}{16}=\frac{139}{560}$, and E has $\frac{13}{420}+\frac{1}{16}=\frac{157}{1680}$; fifth, A has $\frac{1}{4}$ of $\frac{1}{5}=\frac{1}{20}$, B has $\frac{1}{8}+(\frac{1}{3}$ of $\frac{1}{5})=\frac{23}{120}$, C has $\frac{1}{5}-(\frac{1}{20}+\frac{1}{15}+\frac{2}{35})=\frac{11}{420}$, D has $\frac{139}{560}+(\frac{2}{7}$ of $\frac{1}{5})=\frac{171}{560}$, and E has $\frac{157}{1680}$; sixth, A has $\frac{1}{20}-(\frac{3}{4}$ of $\frac{1}{20})=\frac{1}{80}$, B has $\frac{23}{120}-(\frac{3}{4}$ of $\frac{1}{15})=\frac{17}{120}$, C has $\frac{11}{420}$, D has $\frac{171}{560}+\frac{7}{80}=\frac{11}{28}$, and E has $\frac{157}{1680}$; seventh, A has $\frac{1}{80}+\frac{7}{256}=\frac{51}{1280}$, B has

$\frac{17}{120}+\frac{7}{256}=\frac{649}{3840}$, C has $\frac{11}{420}$, D has $\frac{11}{28}-\frac{7}{128}=\frac{303}{896}$, and E has $\frac{157}{1680}$; eight, A has $\frac{51}{1280}-\frac{21}{2048}+\frac{1}{15}=\frac{2957}{30720}$, B has $\frac{649}{3840}-\frac{21}{2048}+\frac{1}{15}=\frac{6925}{30720}$, C has $\frac{11}{420}+\frac{7}{1024}+\frac{1}{15}=\frac{10719}{107520}$, D has $\frac{303}{896}+\frac{7}{1024}+\frac{1}{15}=\frac{44263}{107520}$, and E has $\frac{157}{1680}+\frac{7}{1024}+\frac{1}{15}=\frac{17951}{107520}$, or reducing these fractions to a common denominator, we have the following: A $\frac{20699}{215040}$, B $\frac{48475}{215040}$, C $\frac{21438}{215040}$, D $\frac{88526}{215040}$, E $\frac{35902}{215040}$, the sum of which is $\frac{215040}{215040}=1$ as it should be.

Excellent solutions of this problem were received from *G. B. M. Zerr*, *E. W. Morrell*, and *P. S. Berg*

ERRATUM—In the solution of problem 42, Professor Cooper D. Schmitt's address should read, Professor of Mathematics, University of Tenn. etc.

PROBLEMS.

48. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Fifty thousand days preceding Thursday, March 7, 1895, was what date and what day of the week?

49. Proposed by J. A. CALDERHEAD, B. Sc., Superintendent of Schools, Limaville, Ohio.

I have a garden in the form of an equilateral triangle, whose sides are 200 feet. At each corner stands a tower: the height of the first is 30 feet, the second is 40 feet, and the third is 50 feet. At what distance from the base of each tower must a ladder be placed, that it may just reach the top of each? And what is the length of the ladder, the garden being a horizontal plane?

[From *Greenleaf's National Arithmetic*.]

Give a solution simple enough to be presented to a class in arithmetic.

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

39. Proposed by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Find x, y, z, and w from the equations

$$x^4+y^4+z^4+x^2+y^2+z^2=112\ldots(1),$$
$$x^4+z^4+w^4+x^2+z^2+w^2=382\ldots(2),$$
$$x^4+y^4+w^4+x^2+y^2+w^2=294\ldots(3),$$
$$y^4+z^4+w^4+y^2+z^2+w^2=364\ldots(4).$$

I. Solution by A. H. BELL, Hillsboro, Illinois, P. S. BERG, Apple Creek, Ohio, D. G. DURRANCE, Jr., Camden, N.Y., COOPER D. SCHMITT, A.M., University of Tennessee, and H.C. WILKES, Murraysville, West Virginia.

Adding the four equations and dividing the result by 3, we readily obtain, $x^2+y^2+z^2+w^2+x^4+y^4+z^4+w^4=384$. From this subtract each equation in order and we obtain $w^2+w^4=272$, $y^2+y^4=2$, $z^2+z^4=90$, $x^2+x^4=20$, all of which are bi-quadratics. Solving we find, $w=\pm4$ or $\pm\sqrt{-17}$, $y=\pm1$ or $\pm\sqrt{-2}$, $z=\pm3$ or $\pm\sqrt{-10}$, $x=\pm2$ or $\pm\sqrt{-5}$.

II. **Solution by LEONARD E. DICKSON, M. A.. University of Chicago.**

$$(2)-(1) \text{ gives } w^4+w^2-y^4-y^2=270 \qquad (5)$$
$$(3)-(1) \text{ gives } w^4+w^2-z^4-z^2=182 \qquad (6)$$
$$(4)-(1) \text{ gives } w^4+w^2-x^4-x^2=252 \qquad (7)$$
$$(5)-(6) \text{ gives } z^4+z^2-y^4-y^2=88 \qquad (8)$$
$$(5)-(7) \text{ gives } x^4+x^2-y^4-y^2=18 \qquad (9)$$
$$(1)-(9) \text{ gives } 2y^4+2y^2+z^4+z^2=94 \qquad (10)$$
$$(10)-(8) \text{ gives } 3y^4+3y^2=6 \qquad (11)$$
$$\therefore\ y=\pm1 \text{ or } \pm\sqrt{-2}.$$

From (8) and (11), $z^4+z^2=90$. $\therefore\ z=\pm3$ or $\pm\sqrt{-10}$.
From (9) and (11), $x^4+x^2=20$. $\therefore\ x=\pm2$ or $\pm\sqrt{-5}$.
From (5) and (11), $w^4+w^2=272$. $\therefore\ w=\pm4$ or $\pm\sqrt{-17}$.
Hence there are $4^4=256$ sets of values as solutions.

Also solved by B. F. Burleson, H. W. Draughon, J. H. Drummond, J. K. Ellwood, M. A. Gruber, J. F. W. Scheffer, F. P. Matz, and G. B. M. Zerr.

40. **Proposed by B. F. BURLESON, Oneida Castle, New York.**

Find by quadratics all the possible values for x and y in the equations $x^3+y^3=b=35$, ...(1), and $x^2+y^2=a=13$....(2).

I. **Solution by the PROPOSER.**

From equation (1) $y=\sqrt[3]{(b-x^3)}$....(3). From equation (2) $y=\sqrt{(a-x^2)}$....(4). Equating (3) and (4) and clearing from radicals we obtain, $2x^6-3ax^4-2bx^3+3a^2x^2-(a^3-b^2)=0$....(5). Substituting numerical for literal values in (5), it becomes, $2x^6-39x^4-70x^3+507x^2-972=0$....(6). Factoring (6), $(x^2-5x+6)(x^2-2x-4\frac{1}{2})(2x^2+14x+36)=0$....(7). Thus far finding the six roots of equation (6), it is resolved into finding the roots of the three quadratic equations $x^2-5x=-6$....(8), $x^2-2x=4\frac{1}{2}$....(9), and $2x^2+14x=-36$....(10). Resolving equation (8) for the two values of x in it, and then substituting these values severally in (3) or (4) for the corresponding values of y, we get, $x=2$ or 3, and $y=3$ or 2. In the same way we find from eq. (9), $x=1+\sqrt{5\frac{1}{2}}=3.345208+$ and $y=1-\sqrt{5\frac{1}{2}}=-1.345208+$, or $x=1-\sqrt{5\frac{1}{2}}=-1.345208+$, and $y=1+\sqrt{5\frac{1}{2}}=3.345208+$. From eq. (10) we obtain the imaginary roots, $x=-3\frac{1}{2}+\frac{1}{2}\sqrt{-23}$, and $y=-3\frac{1}{2}-\frac{1}{2}\sqrt{-23}$, $x=-3\frac{1}{2}-\frac{1}{2}\sqrt{-23}$ and $y=-3\frac{1}{2}+\frac{1}{2}\sqrt{-23}$. Thus x and y have six values each and no more, all of which we have found by quadratics.

II. **Solution by J. K. ELLWOOD, A. M., Colfax-School, Pittsburg, Pennsylvania, and J. W. WATSON, Middle Creek. Ohio.**

Let $x+y=p$, $xy=s$. Then the equations become, $ap-sp=b$, $a+2s=p^2$. Eliminating s, $p^3-3ap+2b=0$, or $p^3-39p+70=0$. It seems the literal solution can not be completed by using quadratics. But multiplying p^3-39p

$+70=0$ by p, $p^4-39s^2=-70p$. Adding to both sides $25p^2+49$, we have $p^4-14p^2+49=25p^2-70p+49$, whence $p^2-7=\pm(5p-7)$ and $p=2$, 5, or -7; hence $s=6$, $-4\frac{1}{2}$, or 18. [Or, from $p^3-39p+70=0$, we have $(p-5)(p^2+5p-14)=0$. $\therefore$ $p=5$, 2, or -7, $s=6, -4\frac{1}{2}$, or 18.]

$\therefore$ $x+y=5$, $xy=6$, $\therefore$ $x=3$ or 2, $y=2$ or 3.

$x+y=2$, $xy=-4\frac{1}{2}$, $\therefore$ $x=\frac{1}{2}(2\pm\sqrt{22})$, $y=\frac{1}{2}(2\pm\sqrt{22})$.

$x+y=-7$, $xy=18$, $\therefore$ $x=\frac{1}{2}(-7\pm\sqrt{-23}$, $y=\frac{1}{2}(-7\pm\sqrt{-23})$.

$\therefore$ There are six values for x and six values for y admissible.

III. Comment by H. W. DRAUGHON, Olio, Mississippi.

The problem can not be solved by quadratics as may be shown thus: The resulting literal equation $p^3-3ap+2b=0$ can not be solved by quadratics, and therefore the given equations can not be solved by quadratics. Cubics of this class can be *apparently* solved by quadratics, when they have one commensurable root. Let r be one root of the equation $p^3=3ap-2b$, for instance. Subtracting r^2p from both members gives $p^3-r^2p=(3a-r^2)p-2b$. Obviously both members of this equation can be divided exactly by $p-r$, giving a quadratic equation, but before this subtraction can be made we must find r, which can not be done by quadratics. If we substitute the definite values for a and b we readily complete the solution.

Also solved by *A. H. Bell, P. S. Berg, D. G. Durrance, Jr., H. W. Draughon, F. P. Matz, G. B. M. Zerr, J. F. W. Scheffer, C. D. Schmitt*, and *H. C. Wilkes*.

PROBLEMS.

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50. Proposed by LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago.

Given $b=a\sqrt{-1}\,.\tan\frac{m\pi}{n}$, m being an arbitrary integer, find the simplest *real* relation between a and b.

51. Proposed by J. W. NICHOLSON, LL. D., President and Professor of Mathematics, Louisiana State University and A. and M. College, Baton Rouge, Louisiana.

Solve the equation $x^5+5mx^3+5m^2x+n=0$.

GEOMETRY.

Conducted by B.F.FINKEL, Kidder, Missouri. All Contributions to this department should be sent to him.

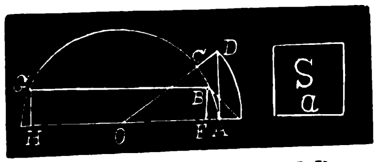
SOLUTIONS OF PROBLEMS.

37. **Proposed by B. F. BURLESON, Oneida Castle, New York.**

Inscribe in a semi-circle (1), a rectangle having a given area: (2), a rectangle having the maximum area.

Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, and S. N. COLLIER, University of Mississippi.

(1). Let ABC be the given semi-circle with center O; and let S be a square of given area with side a.



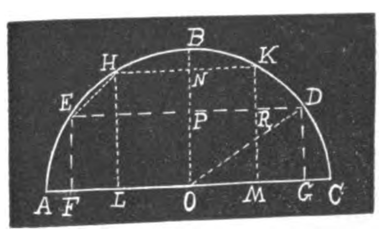
To inscribe in ABC a rectangle equivalent to S.

Construction:—At A, the left-hand extremity of the semi-circumference, draw a tangent AD making it equal to a. Draw OD. With O as center and OD as radius describe an arc cutting OA produced at E. Through E draw a line making an angle of 45° with EO, intersecting the circumference at B. Through B draw a parallel to AO cutting the circumference at G. Through B and G draw perpendiculars to AO meeting the bounding diameter at F and H respectively. Then $FBGH$ is the rectangle required.

Proof:—$OD=\sqrt{(AD^2+OR^2)} = \sqrt{(a^2+R^2)}$, denoting OA by R. But $OD=OE=OF+FE=OF+FB$. $\therefore$ $\sqrt{(a^2+R^2)}=OF+FB$.

Squaring, $a^2+R^2=OF^2+2.OE.FB+FB^2$. But $OF^2+FB^2=R^2$. $\therefore$ $a^2=2.OF.FB$. $2.OF$ is the base of the rectangle and FB is its altitude. Also, a^2=given area. $\therefore$ the rectangle is equivalent to the given area.

(2). Let ABC be a given semi-circle with center O. To inscribe in ABC a maximum rectangle.



Construction:—Draw the radius OD making an angle of 45° with OC. With D as one vertex construct the rectangle $DEFG$. $DEFG$ is the rectangle required.

Proof:—Let $LMKH$ be any other inscribed rectangle; OB the radius perpendicular to OC.

Compare rectangles PG and NM, the halves of the rectangles EG and HM. Rectangle PM is common to the two. Rect. $RG>$ rect. NR; for $DG(=DP)>RP$, and $DR>KR$.

Since $\angle RKD>\angle RDK$, the former being measured by one-half of an arc greater than 90° and the latter by one-half an arc less than 90°.

It follows that $EDGF$ is the rectangle required.

A. L. Foote furnished a neat algebraic solution; *G. B. M. Zerr*, *P. S. Berg*, *Cooper D. Schmitt* solved the problem by the calculus, and *C. D. M. Showalter* gave a good geometrical solution. Space forbids further consideration of this problem.

38. Proposed by LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago.

Give a *strictly geometric* proof of my fundamental theorem of the Inscription of Regular Polygons, viz: Suppose a circle of unit radius divided at the points $A, A_1, A_2, A_3, \ldots. A_p$, ...into $2p+1$ equal parts and the diameter AO drawn. Then, if the chords $OA_1, OA_2, \ldots. OA_p$ be drawn, we have $OA_1 - OA_2 + OA_3 - OA_4 + OA_5 - \ldots. \pm OA_p = 1$.

Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

For plainness we will solve this rpoblem in full for the 13-gon. A general solution is as easy, but not as clearly understood.

Let OA_6, OA_5, OA_4, &c.$=a_1, a_2, a_3$, &c.

$$A_4A_5 = A_5A_6 = A_6A_7 = \&c = c,\ A_4A_6 = A_5A_7 = A_6A_8 = \&c. = d.$$

Now by Ptolemy's Theorem:—The rectangle contained by the diagonals of a quadrilateral inscribed in a circle &c., we easily get the following relations:

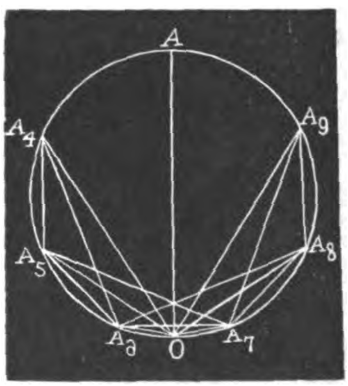
$$\begin{array}{l|l|l}
c(a_1+a_3)=da_2 & c(a_6+a_8)=da_7 & c(a_{10}+a_{12})=da_{11} \\
c(a_2+a_4)=da_3 & c(a_7+a_9)=da_8 & c(a_{11}+a_{13})=da_{12} \\
c(a_3+a_5)=da_4 & c(a_8+a_{10})=da_9 & c(a_{12}-a_1)=da_{13} \\
c(a_4+a_6)=da_5 & c(a_9+a_{11})=da_{10} & c(a_2-a_{13})=da_1 \\
c(a_5+a_7)=da_6 & &
\end{array}$$

Hence $d\{(a_1+a_3+a_5+a_7+a_9+a_{11}+a_{13})-(a_2+a_4+a_6+a_8+a_{10}+a_{12})\}$
$=c\{(a_2-a_{13})+(a_2+a_4)+(a_4+a_6)+(a_6+a_8)+(a_8+a_{10})+(a_{10}+a_{12})+(a_{12}-a_1)$
$-(a_1+a_3)-(a_3+a_5)-(a_5+a_7)-(a_7+a_9)-(a_9+a_{11})-(a_{11}+a_{13})\}$.

$\therefore\ (d+2c)\{(a_2+a_4+a_6+a_8+a_{10}+a_{12})$
$-(a_1+a_3+a_5+a_7+a_9+a_{11}+a_{13})\}=0$.

$\therefore\ a_2+a_4+a_6+a_8+a_{10}+a_{12}$
$=a_1+a_3+a_5+a_7+a_9+a_{11}+a_{13}$.

Generally $a_2+a_4+a_6+\ldots.+a_{2p}$
$=a_1+a_3+a_5+\ldots.+a_{2p+1}$.



In the above, O can be any point between OA_p and OA_{p+1}.

In the problem $OA=a_7=2$, $a_6=a_8$, $a_4=a_6$, $a_2=a_4$, $a_1=a_{13}$, $a_3=a_{11}$, $a_5=a_9$.

$\therefore\ a_2+a_4+a_6=a_1+a_3+a_5+1$, but $a_6=OA_1$, $a_5=OA_2$, $a_4=OA_3$, $a_3=OA_4$, $a_2=OA_5$, $a_1=OA_6$.

$\therefore\ OA_1-OA_2+OA_3-OA_4+OA_5$
$-OA_6=1$, p, even.

For 15-gon. $OA_1-OA_2+OA_3-OA_4+OA_5-OA_6+OA_7=1$, p, odd.

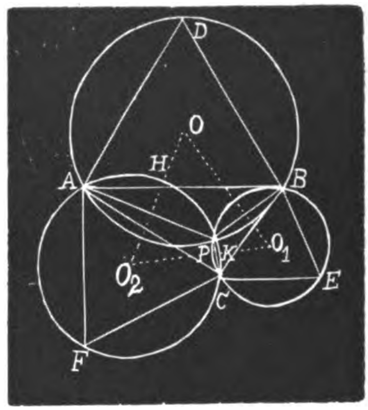
$\therefore\ OA_1-OA_2+OA_3-OA_4+OA_5-\ldots.\pm OA_p=1$, as p is odd or even.

39. Proposed by J. K. ELLWOOD, Principal of Colfax Schools, Pittsburg, Pennsylvania.

If on the three sides of any plane triangle equilateral triangles be described, the lines joining the centres of these equilateral triangles form an equilateral triangle.

I. Solution by F. E. MILLER, Ph. D., Professor of Mathematics, Otterbein University, Westerville, Ohio; Professor J. F. W. SCHEFFER, A. M., Hagerstown, Maryland; JOHN T. FAIRCHILD, Ada, Ohio; J. C. CORBIN, Pine Bluff, Arkansas; and the PROPOSER.

Let ABC be any plane $\triangle$, O, O_1, and O_2 the centres of the equilateral $\triangle$'s constructed. About these $\triangle$'s pass circumferences. They will intersect in a point, P.



Let P be the intersection of the 2 circles, AFC and CEB. Join AP, CP and BP. Since $APCF$ is inscribed, $\angle F+\angle APC=180^\circ$. But $\angle F=60^\circ$. $\therefore$ $\angle APC=120^\circ$. Similarly, $\angle CPB=120^\circ$. $\therefore$ $\angle APB=120^\circ$; and $\angle APB+\angle D=180^\circ$. $\therefore$ $APBD$ is inscribed, and P is in the circumference of DAB. (Q. E. D.)

Lines that join the centres of intersecting circles bisect the common chords and the intercepted arcs. $\therefore$ arc $HP=\frac{1}{2}$ arc AP; and arc $PK=\frac{1}{2}$ arc PC. $\therefore$ arc $HPK=\frac{1}{2}$ arc APC. But arc APC measures the angle $F=60^\circ$ at circumference; therefore its half HK measures an equal angle at the centre.

$\therefore$ $\angle O_2=\angle F=60^\circ$. Similarly, $\angle O_1$ may be shown $=\angle E$, and $\angle O=\angle D$. But equiangular $\triangle$'s are also equilateral. $\therefore$ OO_1O_2 is equilateral.

Q. E. D.

II. Solution by Professor G. B. M ZERR, A. M, Principal of High School, Staunton, Virginia, and Professor J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

Let O, O_1, O_2 (Fig. in Solution I.) be the centres of the equilateral triangles on the sides BA, BC, AC respectively, of the triangle ABC. Let Δ=area ABC.

Then $\angle OAB=\angle OBA=\angle O_1BC=\angle O_1CB=\angle O_2CA=\angle O_2AC=30^\circ$.

$$OB=OA=\frac{c}{\sqrt{3}},\quad O_1B=O_1C=\frac{b}{\sqrt{3}},\quad O_2C=O_2A=\frac{a}{\sqrt{3}}.$$

$$\therefore\quad OO_1=\sqrt{\frac{c^2}{3}+\frac{b^2}{3}-\frac{2bc}{3}\cos.(60^\circ+A)}$$

$$=\sqrt{\frac{c^2}{3}+\frac{b^2}{3}-\frac{1}{3}bc\cos.A+\frac{\sqrt{3}}{3}bc\sin A}$$

$$=\sqrt{\frac{1}{6}(a^2+b^2+c^2+4\sqrt{3}\Delta)}=O_1O_2=O_2O$$

$\therefore$ OO_1O_2 is equilateral.

Excellent solutions of this problem were also received from *Professors F. P. Matz* and *G. I. Hopkins*.

PROBLEMS.

44. Proposed by I. J. SCHWATT, Ph. D., Professor of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania.

(1). If from the middle point M of the side BC of the triangle ABC a parallel to the bisector AF of the external angle to ABC is drawn to meet AB at K, the point K divides then the side AB in KA

$$=\tfrac{1}{2}(AB+AC) \text{ and } KB=\tfrac{1}{2}(AB-AC).$$

(2). If K is joined to the extremity D of the diameter perpendicular to BC then is KD perpendicular to AB.

45. Proposed by B. F. BURLESON, Oneida Castle, New York.

Determine the radius of a circle circumscribing three tangent circles of radii $a=15$, $b=17$, and $c=19$.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

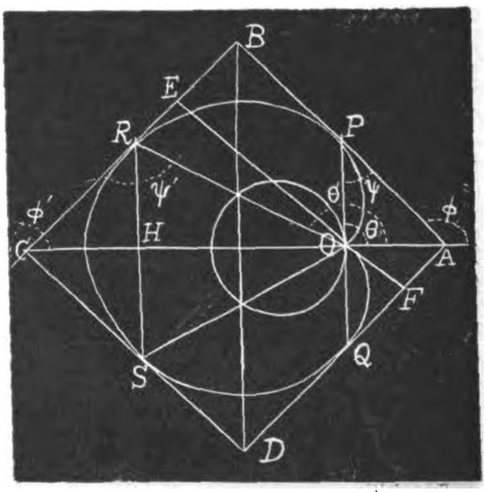
SOLUTIONS OF PROBLEMS.

30. Proposed by E. W. NICHOLS, Professor of Mathematics in the Virginia Military Institute, Lexington, Virginia.

Given the cardioid $r=a\,(1-\cos\theta)$: find the area of its circumscribing square formed by tangents making angles of 45° with its axis.

I. Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let $OPRSQ$ be the cardioid. Draw PQ through the cusp perpendicular to the initial line AC. From the property of the cardioid the angle APO, made by the tangent and radius vector at P, $=\frac{1}{2}\angle POA$. But $\angle AOP=\frac{1}{2}\pi$. $\therefore\ \angle OPA=\angle OAP=\frac{1}{4}\pi$. $\therefore$ the tangents BA, DA at the points P, Q are inclined at an angle of 45° to the axis and are perpendicular to each other. Draw the radii vectors OR, OS, making the $\angle ROP=\angle SOQ=\frac{1}{3}\pi$, and draw the tangents CB, CD at the points R, S. Then $\angle ROP=\angle SOQ=\frac{1}{3}\pi$, $\angle OPB=\angle OQD=\frac{3}{4}\pi$, $\angle ORB=\angle OSD=\frac{5}{12}\pi$.



$\therefore\ \angle ROP+\angle OPB+\angle ORB$
$=\angle SOQ+\angle OQD+\angle OSD=\frac{3}{2}\pi.$

$\therefore\ \angle B=\angle D=\frac{1}{2}\pi$, an $ABCD$ is the required square.

Now $OR=a(1-\cos\frac{5}{6}\pi)=a(1+\frac{1}{2}\sqrt{3})$, $OQ=a(1-\cos\frac{3}{2}\pi)=a$.

$$OE=OR\sin ORB=OR\sin 75^\circ=\frac{a(2+\sqrt{3})}{2}\cdot\frac{1+\sqrt{3}}{2\sqrt{2}}=\frac{5+3\sqrt{3}}{4\sqrt{2}}a.$$

$$OF=OQ\sin OQF=OQ\sin 45^\circ=\frac{a}{\sqrt{2}}.$$

$$AB=OE+OF=\frac{3(3+\sqrt{3})}{4\sqrt{2}}a.$$

Area square $=AB^2=\dfrac{27(2+\sqrt{3})}{16}a^2$. A solution without the use of Calculus.

II. **Solution by Cadet A. R. GATEWOOD, Virginia Military Institute, Lexington, Virginia; and COOPER D. SCHMITT, A. M., University of Tennessee, Knoxville, Tennessee.**

$$\text{Tan}\,\Psi=r\Big/\frac{dr}{d\theta}\cdot\frac{dr}{d\theta}=a\sin\theta.\quad\therefore\ \tan\Psi=\frac{r}{a\sin\theta}=\frac{2a\sin^2\frac{\theta}{2}}{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$=\tan\frac{\theta}{2}.\quad\therefore\ \Psi=\frac{\theta}{2}.\quad \phi=\Psi+\theta=\frac{\theta}{2}+\theta=\frac{3\theta}{2}.$$

Now when tangents make angles of 45° each with the initial line, $\phi=135^\circ$, $\phi'=225^\circ$. $\therefore$ $\theta=90^\circ$, $\theta'=150^\circ$, $\angle ROH=30^\circ$.

When $\theta=90^\circ$, $OA=OP=a$. When $\theta=150^\circ$, $RO=a(1-\cos150^\circ)=a\left(1+\dfrac{\sqrt{3}}{2}\right)$,

$CH=RH=RO\sin30^\circ=\frac{1}{2}a\left(1+\dfrac{\sqrt{3}}{2}\right)$, $HO=RO\cos30^\circ=\dfrac{\sqrt{3}}{2}a\left(1+\dfrac{\sqrt{3}}{2}\right)$.

Now $CA=CH+HO+OA=\frac{1}{2}a\left(1+\dfrac{\sqrt{3}}{2}\right)+\dfrac{\sqrt{3}}{2}a\left(1+\dfrac{\sqrt{3}}{2}\right)+a=\dfrac{3}{4}a(3+\sqrt{3})$.

$ABCD=BA^2=\frac{1}{2}CA^2=\frac{1}{2}\left[\dfrac{3}{4}a(3+\sqrt{3})\right]^2=\frac{27}{16}(2+\sqrt{3})a^2$, which is the area of the circumscribed square.

III. **Solution by F. P. MATZ, Ph. D., New Windsor College, New Windsor, Maryland; ALFRED HUME, C. E., University of Mississippi, P. O., Mississippi; and J. SCHEFFER, A. M., Hagerstown, Maryland.**

Since the sides of the circumscribing square are to intersect the axis at angles of 45°, we have from *Todhunter's Differential Calculus*, p. 304, Art. 278, that

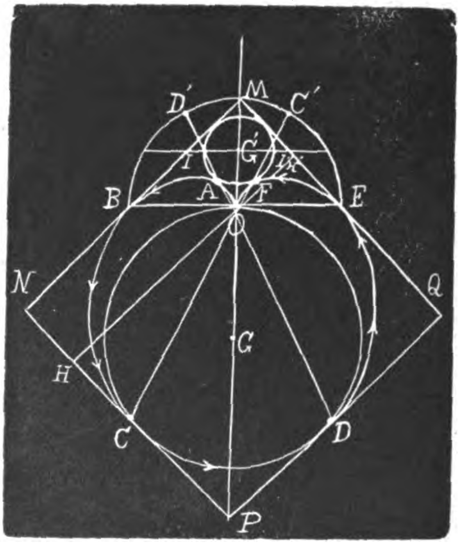
$$\tan\Psi=\frac{dy}{dx}=\frac{\sin\theta(dr/d\theta)+r\cos\theta}{\cos\theta(dr/d\theta)-r\sin\theta},$$

$$=\frac{\sin\theta(a\sin\theta)+a(1-\cos\theta)\cos\theta}{\cos\theta(a\sin\theta)-a(1-\cos\theta)\sin\theta}=\pm1\ldots(1).$$

From (1) we have, respectively:

$\tan\frac{3}{2}\theta=+1$. $\therefore$ $\theta=\frac{1}{6}\pi$, $\frac{5}{6}\pi$, $\frac{9}{6}\pi$, etc., and $\tan\frac{3}{2}\theta=-1$. $\therefore$ $\theta=\frac{3}{6}\pi$, $\frac{7}{6}\pi$, $\frac{11}{6}\pi$, etc.

Taken in the order of their magnitude, these values of θ represent the angular position of the points A, B, C, D, E, and F, with respect to the origin of polar co-ordinates and the axis of the cardioid. If any tangents to the cardioid be drawn through *these* points, such tangents make angles of 45° with the axis MP. When the origin of polar co-ordinates is at O, the radius vectors of the points already specified becomes respectively:



$OA=\frac{1}{2}(2-\sqrt{3})a$, $OB=a$,
$OC=\frac{1}{2}(2+\sqrt{3})a$, $OD=\frac{1}{2}(2+\sqrt{3})a$,
$OE=a$, and $OF=\frac{1}{2}(2-\sqrt{3})a$.

Consequently the area of the required circumscribing square
$=(MN)^2=(OH+OH')^2$
$=\frac{9}{32}(3+\sqrt{3})^2a, =\frac{27}{16}(2+\sqrt{3})a^2\ \dots.(2)$;
and of this square, the diagonal MP $=\frac{3}{4}(3+\sqrt{3})a$. Since the *diagonal* $II'=\frac{3}{4}(3+\sqrt{3})a$, the area of the square the center of the inscribed circle of which is at $G'=(MI)^2=\frac{9}{32}(3-\sqrt{3})^2a^2$, $=\frac{27}{16}(2-\sqrt{3})a^2\ \dots.(2)$.

Represent the area of the *larger* square by A and that of the *smaller* square by A'; then from the results given, $A:A'::(2+\sqrt{3}):(2-\sqrt{3})::OC:OF$.

IV. Second Solution by Professors F. P. MATZ; and C. E. WHITE, Trafalgar, Indiana.

The *pedal equation* of the cardioid in consideration, is $p^2=r^3/2a$; that is, for the points B, C, D, and E, we have respectively: $p_B=a/\sqrt{2}$, $p_C=\frac{1}{4}a\sqrt{(26+15\sqrt{3})}$, $p_D=\frac{1}{4}a\sqrt{(26+15\sqrt{3})}$, and $p_E=a/\sqrt{2}$. In order that the quadrilateral circumscribing the cardioid may be a square, we must have $p_B+p_D=p_C+p_E$; and this condition is fulfilled. Hence the *required* area becomes $A=(p_B+p_D)^2=(p_C+p_E)^2=\frac{27}{16}(2+\sqrt{3})a^2\ \dots.(1)$,

while the area of the *smaller* square becomes

$$A'=(p_B-p_F)^2=(p_E-p_A)^2=\frac{27}{16}(2-\sqrt{3})a^2\ \dots.(2).$$

NOTE—By using the equation $p=2a\sin^3\phi$, in which $\phi=\frac{1}{3}\theta$, a *third* solution can be made. Sufficient data are given in the problem to enable us to make a *fourth* solution, without having recourse to the differential calculus.

Also solved by *O. W. Anthony*, *H. W. Draughon*, and *J. B. Faught*. We regret that *Professor Faught's* solution was mislaid and could not be considered in selecting papers for publication.

PROBLEMS.

41. Proposed by F. P. MATZ, M. Sc., Ph.D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

The *closed portion* of the curve known as "The Cocked Hat," equation

$$x^4+x^2y^2+4ax^2y-2a^2x^2+3a^2y^2-4a^3y+a^4=0,$$

revolves around the axis of y. Find the *companulate* volume generated. If the same portion of the curve revolve around the axis of x, find the *fusiform* volume generated. Also, determine the area of this closed portion of the curve.

41. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

A railroad turn-table 100 feet long is balanced upon a pivot in the center of a circular track 100 feet in diameter. How far does a man walk who starts at one end of the table and walks, at a uniform rate, the entire length of the table in the same time that the table makes two revolutions, if the table starts to turn at the same time the man starts to walk?

MECHANICS.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

16. Proposed by A. H. BELL, Hillsboro, Illinois.

An iron bar 20 feet long and weighing 2,000 lbs. leans against a wall at angles of 30°, 45°, and 80°. Determine the *pressure* upon the floor, and *that* upon the wall.

I. Solution by F. P. MATZ, M. Sc., Ph.D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Let P_F=the pressure upon the floor, and P_W=the pressure upon the wall, then, if θ be the angle the bar makes with ths vertical wall, we have from *Wood's Analytical Mechanics*, the equations:

$P_F=-W=-2000$ lbs; that is, the floor sustains the whole weight of the bar, and $P_W=\frac{1}{2}W\tan\theta$....(1).

According to the conditions of the problem, we deduce from (1) the following results:

For θ	= 1°,	we have	P_W=	17.455054+lbs;
" "	=10°,	" "	" =	176.327696+ " ;
" "	=30°,	" "	" =	577.350266+ " ;
" "	=45°,	" "	" =	1000.000000+ " ;
" "	=60°,	" "	" =	1732.050800+ " ;
" "	=80°,	" "	" =	5671;280256+ " ;
" "	=89°,	" "	" =	5728.996052+ " ;

NOTE.—How the *last two* results, and all *succeeding* results to the limit of the quadrant, are to be interpreted, is a question on which Professor DeVolson Wood can *interest* the readers of the MONTHLY.

II. Solution by EDMOND FISH, Hillsboro, Illinois.

It must be assumed that the bar is prevented from sliding down by some lateral resistance, either the roughness of the floor, or some object, as a rod or plank laid between the foot of the bar and an opposite wall. Now let a rope be passed from the foot of the bar over a pulley. What weight attached to the rope will relieve the floor of all pressure?

By the Law of Virtual Velocities, Power and Weight are inversely proportioned to their rates of movement in the direction required. In the present case these rates are obviously equal, and the power must equal the weight. Hence the pressure on the floor will be 2000 lbs. for all positions of the bar.

For the pressure on the wall, suppose the bar inclined 30° from the vertical. Its whole weight may be considered concentrated in its center of gravity. Its tendency to move is in an arc whose radius is 10. But only the vertical part of this movement is effective, and this part is expressed by $\sin 30^\circ$.

This supposed movement of the center of the bar implies a movement of the upper end in an arc whose radius is 20. But we are concerned only with the horizontal part of this movement and this is expressed by $\cos 30^\circ$. Hence, by the law before quoted, Power (2000 lbs): Weight :: $2\cos 30^\circ : \sin 30^\circ$. (*Twice* $\cos 30^\circ$ because R is double).

$$\therefore \text{ Weight or pressure on wall} = \frac{2000 \times \sin 30}{2 \cos 30} = 1000 \times \tan 30^\circ.$$

In general the pressure on the wall equals half the weight of the bar multiplied by the tangent of the inclination.

At 30° A is $577\tfrac{35}{100}$ lbs.
At 45° A is 1000 lbs.
At 80° A is $5671\tfrac{28}{100}$ lbs.

This problem was also solved by *P. S. Berg and J. F. W. Scheffer*.

17. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Find the law of density of strings collected into a heap at the edge of a table with the end of the string just over the edge, so that equal masses may always pass over in equal units of times.

Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

A pulverized solid, if piled up, will settle by the force of gravity to a certain inclination, according to the smallness and smoothness of its particles. Now, a string is practically a cylindric solid of *great* length, *small* cross-sections, and *indefinite* flexibility and compresibility. In passing over the edge of the table, the string will pile into an *approximate* right circular cone. A stream of pulverized solid—an *impalpable powder*, like precipitated $BaSO_4$—is practically a *string* of indefinitely small molecular attraction; and such a string will pile into a right circular cone.

Let $AO=r_1$, $CI=m_1$, $\angle CAO=\omega_1$ and δ_0=the initial density of the first *string-cone* formed. Pass a pound (W_1) of string *uniformly* over the edge of the table; then from the cone $AOB-C$, we have

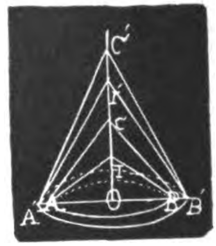
$$W_1 = M_1 g = V_1\delta_0, = \tfrac{1}{3}\pi r_1^3 \delta_0 g \tan\omega_1 \ldots . (1).$$

Pass similarly a second pound of string; then will be formed the cone

$A'OB'-C'$, and the cone $AOB-C$ will be compressed into the cone $AOB-I$.

The volume of the *compressed* string-cone, $AOB-I$, becomes

$W_1=M_1g=V_1\delta_1g,=\frac{1}{3}\pi r_1^2(r_1\tan\omega_1-m_1)\delta_1 g\ldots.(2).$



Equating the right-hand members of (1) and (2), we have

$\delta_1=\left(\frac{r_1\tan\omega_1}{r_1\tan\omega_1-m_1}\right)\delta_0,\ldots,\delta_n=\left(\frac{r_1\tan\omega_1}{r_1\tan\omega_1-m_n}\right)$ $\delta_1\ldots.(3).$

The values of $\delta_1\ldots.\delta_n$, as determined empirically for determinate conditions, immediately lead to the required law of density.

18. **Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Mississippi.**

An elleptic paraboloid whose equations is $\frac{y^2}{a}+\frac{x^2}{b}=2z$ has its axis vertical and vertex downward. If μ be the co-efficient of friction, prove that a heavy particle will rest at any point of the surface below its intersection with the cylinder $\frac{y^2}{a^2}+\frac{x^2}{b^2}=\mu^2$.

Solution by the PROPOSER.

If W is the weight of the particle, N and T its normal and tangential components, $W^2=N^2+T^2$. Also, when the particle is on the point of sliding, $T=\mu N$. Hence, $W^2=(1+\mu^2)N^2$. Again, $W\cos\theta=N$, θ being the angle between the normal and the Z-axis.

Now $\cos\theta=\dfrac{\frac{dF}{dz}}{\sqrt{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}\right)^2}}$, $F(x,y,z)=0$ being the equation of the surface, and the differential-coefficients being partial.

$\frac{dF}{dx}=\frac{2x}{b},\frac{dF}{dy}=\frac{2y}{a},\frac{dF}{dz}=-2.$ Substituting, $\cos\theta=\dfrac{-2}{\sqrt{\left(\frac{4x^2}{b^2}+\frac{4y^2}{a^2}+4\right)}}$

This, in the fourth equation above, gives, after squaring,

$N^2=\dfrac{4W^2}{\frac{4x^2}{b^2}+\frac{4y^2}{a^2}+4}$. Substituting this value of N^2 in the third equation and reducing we get $\frac{y^2}{a^2}+\frac{x^2}{b^2}=\mu^2$.

This is the relation between the x and y co-ordinates of every point of the surface at which the friction is limiting; in other words, these points lie on the cylindrical surface of which this is the equation. Consequently, their locus is the curve of intersection of the paraboloid and the cylinder. This curve divides the given surface into two parts. The particle will be in equilibrium at any point of the lower and at no point of the upper.

Two excellent solutions of this problem were received from *F. P. Matz*, and one from *G. B. M. Zerr*.

PROBLEMS.

26. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

If an elastic sphere be electrified in such a manner that the initial internal pressure remains constant, determine an expression for the *ratio of the electrical densities* when the volume of the sphere has been increased to $(m+1)$ times its initial volume.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

20. Proposed by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Find two integral numbers, whose sum, difference, and difference of their squares shall be a square, cube, and fourth power.

I. Solution by J. H. DRUMMOND; H. C. WILKES; and M. A. GRUBER.

Let x and y=the two integral numbers. Any number to be a square, a cube, and a fourth power, must also be a twelfth power.

$$\text{Then } x+y=a^{12}$$
$$x-y=b^{12}$$
$$x^2-y^2=a^{12}b^{12}=(ab)^{12}.$$

$$\text{Whence } x=\tfrac{1}{2}(a^{12}+b^{12}), \text{ and } y=\tfrac{1}{2}(a^{12}-b^{12}).$$

In order that x and y be integral, a^{12} and b^{12} must be both odd or both even.

Put $a^{12}=5^{12}=244,140,625$ and $b^{12}=3^{12}=531441$. Then $x=122,336,033$ and $y=121,804,592$. Put $a^{12}=6^{12}=2,176,782,336$ and $b^{12}=2^{12}=4096$. Then $x=1,088,393,216$ and $y=1,088,389,120$.

The lowest values of x and y are found by putting

$$x+y=a^{12}=3^{12}=531441.$$
$$x-y=b^{12}=1^{12}=1.$$

$$\text{Whence } x=265721 \text{ and } y=265720.$$

Many answers can be obtained but the work will be tedious.

II. **Solution by the PROPOSER.**

Let, $x^m+\frac{m(m-1)}{2}x^{m-2}y^2+\ldots+\frac{m(m-1)}{2}x^2y^{m-2}+y^m$, and $mx^{m-1}y+\frac{m(m-1)(m-2)}{2.3}x^{m-3}y^3+\ldots+\frac{m(m-1)(m-2)}{2.3}x^3y^{m-3}+mxy^{m-1}$ be the numbers.

Then $(x+y)^m$, $(x-y)^m$, $(x^2-y^2)^m$, is their sum, their difference, and the difference of their squares. m must be divisible by 2, 3, and 4; this is the case when $m=12$. Then the numbers are $x^{12}+66x^{10}y^2+\ldots+66x^2y^{10}+y^{12}$, and $12x^{11}y+220x^9y^3+\ldots+220x^3y^9+12xy^{11}$ and $(x+y)^{12}$, $(x-y)^{12}$, $(x^2-y^2)^{12}$ is their sum, their difference, and the difference of their squares.

Let $x=2$, $y=1$, then the numbers are 265721, 265720. Their sum $=(3)^{12}=531441=(729)^2=(81)^3=(27)^4$. Their difference $=1=(1)^2=(1)^3=(1)^4$. Difference of their squares $=531441=(729)^2=(81)^3=(27)^4$.

Many other numbers can be found satisfying the conditions.

Also solved by *H. W. Draughon*, and *J. F. W. Scheffer*.

21. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

Find (1) nine positive *integral numbers* in arithmetical progression the sum of whose squares is a *square number*; and (2) find nine integral *square numbers* whose sum is a *square number*.

I. **Solution by H. W. DRAUGHON, Olio, Mississippi.**

(1). Let, $x+4y$, $x+3y$, $x+2y$, $x+y$, x, $x-y$, $x-2y$, $x-3y$, and $x-4y$ be the numbers. Then we are to make the sum of their squares, $9x^2+60y^2=\square$. Let us assume, $9x^2+60y^2=(3x+6m)^2=9x^2+36mx+36m^2$; then,

$x=\frac{60y^2-36m^2}{36m}=\frac{5y^2}{3m}-m$. In order that x may be integral put, $y=3pm$...(1).

Then, $x=15p^2m-m=(15p^2-1)m$....(2). p and m can have any positive, integral values that will make $x>4y$.

Let us make, $p=1$ and $m=1$; then, $x=14$, $y=3$, and the numbers are, 2, 5, , 11, 14, 17, 20, 23, and 26. The sum of their squares is $(48)^2$.

Again, let, $p=2$, and $m=1$; then, $x=59$, $y=12$, and the numbers are, 11, 23, 35, 47, 59, 71, 83, 95, and 107. The sum of the squares of this set is $(183)^2$. An infinite number of sets can be thus obtained from (2).

(2). In the *Mathematical Messenger*, Vol. 7, No. 5. page 47, I find the following formula for n square numbers whose sum is a square:

$S+\frac{1}{4}(p-q)^2=\frac{1}{4}(p+q)^2$, in which $S=pq=$the sum of $n-1$ square numbers. Here, $n=9$. Let us assume, $S=(3)^2+(4)^2+(5)^2+(8)^2+(9)^2+(10)^2+(11)^2+(12)^2=560=10\times56$. Let us make $p=56$ and $q=10$; then, we have, $S+(23)^2=(33)^2$. Any other factors of 560 may be taken. When factors give fractional results we clear of fractions.

II. **Solution by R. J. ADCOCK, Larchland, Illinois.**

$a, a+x, a+2x, a+3x, \ldots a+(n-1)x$, is an arithmetical progression of n terms, their sum of squares is $a^2+(a+x)^2+(a+2x)^2+(a+3x)^2+\&c.\ldots$ $+[a+(n-1)x]^2=S=na^2+(n^2-n)ax+\frac{1}{6}(2n^3-3n^2+n)x^2$ by method of difference. This sum is made=a rational square by the method given in Encyclopedia Britannica, Vol. I, Algebra article 121, 9th edition. When n is a square number $na^2+(n^2-n)ax+\frac{1}{6}(2n^3-3n^2+n)x^2=(n^{\frac{1}{2}}a+hx)^2$

$=na^2+2ahn^{\frac{1}{2}}x+\frac{1}{6}(2n^3-3n^2+n)x^2$. Then $x=\dfrac{2ahn^{\frac{1}{2}}-(n^2-n)a}{\frac{1}{6}(2n^3-3n^2+n)-h^2}=\frac{3}{2}$ for $a=1$, $n=9$, $h=14$. The algebraic value of x in expressions for S_1 after clearing denominatives gives a general solution, the numerical $\frac{3}{2}$, gives $2^2+5^2+8^2+11^2+14^2+17^2+20^2+23^2+26^2=48^2$.

Also solved by *F. P. Matz. G. B. M. Zerr. J. H. Drummond. C. D. Schmitt. J. Scheffer. and M. A. Gruber*

PROBLEMS.

30. Proposed by COOPER D. SCHMITT, Knoxville, Tennessee.

A and B are two integers, A consisting of $2m$ figures each being 1, and B consisting of m figures each being 4. Prove that $A+B+1$ is a square.

31. Proposed by M. A. GRUBER, War Department, Washington, D. C.

How many scalene triangles, of integral sides, can be formed with an altitude of 12? How many isosceles triangles?

AVERAGE AND PROBABILITY.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

17. Proposed by A. L. FOOTE, No. 80, Broad St. New York.

A person 30 years of age has an annuity for 10 years, the present worth of which is \$1000, provided he lives but ten years; for, if he dies, the annuity ceases What is the annuity worth, on the assumption that 75 out of every 4385 persons die annually, between the ages 30 and 40 years?

Solution by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Kidder Institute, Kidder, Missouri.

Let S=the annuity, the present value, P, of which, for $10,=n$, years is \$100.

Then $P=\frac{S}{R-1}\left(\frac{R^n-1}{R^n}\right)$, whence $S=\frac{rPR^n}{R^n-1}$, where $R=1+r$. If we assume $r=.0$, we find $S=\$149.0294$ nearly.

The limit of B's life is $4385\div75=58\frac{7}{15}$ years, $=l$. $\therefore$ the probability that B will be living at the end of 1 year is $\frac{l-1}{l}$; at the end of two years, $\frac{l-2}{l}$; at the end of three years, $\frac{l-3}{l}$; etc.

The present worth of $S, =\$149.20207$, due in 1, 2, 3, 4, etc., years is $\frac{S}{R}$, $\frac{S}{R^2}$, $\frac{S}{R^3}$, etc., to $\frac{S}{R^n}$. The present worth of S due at the end of any year multiplied by the probability of B's living to the end of that year is the actual value of S.

$$\therefore\ S'=\frac{S(l-1)}{Rl}+\frac{S(l-2)}{lR^2}+\frac{S(l-3)}{lR^3}+\ldots\ldots\frac{S(l-n)}{lR^n}\ldots\ldots(1).$$

$$\frac{S'}{R}=\frac{S(l-1)}{lR^2}+\frac{S(l-2)}{lR^3}+\frac{S(l-3)}{lR^4}\ldots\ldots\frac{S(l-n)}{lR^{n+1}}\ldots\ldots(2),\text{ by multiplying (1) by }\frac{1}{R}.$$

$$\therefore\ S'\left(\frac{R-1}{R}\right)=\frac{S(l-1)}{Rl}-\frac{S}{lR^2}-\frac{S}{lR^3}-\ldots\ldots-\frac{S(l-n)}{lR^{n+1}},$$

$$=\frac{S}{l}\left\{\frac{l}{R}-\frac{1}{R}-\frac{1}{R^2}-\frac{1}{R^3}\ldots\ldots\frac{1}{R^n}-\frac{l-n}{R^{n+1}}\right\},$$

$$=\frac{S}{l}\left\{\frac{l}{R}-\frac{l-n}{R^{n+1}}-\frac{1}{R}-\frac{1}{R^2}\ldots\ldots\frac{1}{R^n}\right\},$$

$$=\frac{S}{l}\left\{\frac{l}{R}-\frac{l-n}{R^{n+1}}-\frac{1}{R}\left[1-\frac{1}{R^n}\right]\div\frac{R-1}{R}\right\},$$

$$=\frac{S}{R}\left\{\left(1-\frac{l-n}{lR^n}\right)-R\left(1-\frac{1}{R^n}\right)\div(R-1)l\right\},\text{ whence }S'=S/r\{[1-(l-n)/lR^n]-R(1-1/R^n)/rl\},=\$911.881029.$$

No solutions of this problem were received from our contributors.

18. **Proposed by H. W. DRAUGHON, Clinton, Louisiana.**

The probability that A will speak the truth is twice the probability that B will, in an independent statement, speak the truth: but, if A exerts his influence, the probability is that B will agree with him in any statement. What is the probability of the truth of their concurrent testimony, the chances being equal that A may or may not be interested in the matter?

Solution by P. H. PHILBRICK, C. E., Lake Charles, Louisiana.

1. Suppose that A is not interested in the matter. Let x=the probability of the truth of any one of B's statements. Then $2x$=the probability of the truth of any one of A's statements. The event did occur if both witnesses tell the truth the probability of which is $x\times2x=2x^2$.

The event did not occur if both witnesses testify falsely the probability of which is $(1-x)(1-2x)$. Hence the probability of the occurrence of the event supposing x to be known is, $p'=\dfrac{2x^2}{2x^2+(1-x)(1-2x)}$.

Now as the veracity of A may vary from 0 to 1 x, may vary from 0 to $\frac{1}{2}$, and, therefore, the required probability is

$$p=\int_0^{\frac{1}{2}}p'dx\div\int_0^{\frac{1}{2}}dx=4\int_0^{\frac{1}{2}}\frac{x^2dx}{2x^2+(1-x)(1-2x)}=64\int_0^{\frac{1}{2}}\frac{x^2dx}{(8x-3)^2+7}.$$

Let $8x-3=y$, then $x=\frac{1}{8}(y+3)$, $dx=\frac{1}{8}dy$ and the limits of y are 1 and -3.

Hence, $p=\frac{1}{8}\int_{-3}^{+1}\frac{(y+3)^2dy}{y^2+7}=\frac{1}{8}\int_{-3}^{+1}\left(1+\frac{6y}{y^2+7}+\frac{2}{y^2+7}\right)dy$

$$=\left[\tfrac{1}{8}y+\tfrac{3}{8}\log_e(y^2+7)+\frac{1}{4\sqrt{7}}\tan^{-1}\frac{y}{\sqrt{7}}\right]_{-3}^{1}=\tfrac{1}{2}-\tfrac{3}{8}\log_e 2+\frac{1}{4\sqrt{7}}\tan^{-1}\sqrt{7}.$$

2. Suppose that A is interested in the matter. In this case B's testimony agrees with that of A, and A's alone is to be considered. The probability of the truth of A's testimoney is $P'=2x$ in which x may vary from 0 to $\frac{1}{2}$. Hence, the required probability is $P=\int_0^{\frac{1}{2}}P'dx\div\int_0^{\frac{1}{2}}dx=\left[2x^2\right]_0^{\frac{1}{2}}=\frac{1}{2}$.

Since the chances are equal that A may be or may not be interested in the matter, the probability required is equal to the half sum of the preceeding results, or $p_1=\frac{1}{2}\left(1-\frac{3}{8}\log_e 2+\frac{1}{4\sqrt{7}}\tan^{-1}\sqrt{7}\right)$.

PROBLEMS.

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28. Propsed by G. I. HOPKINS, Instructor in Mathematics and Physics in High School, Manchester, N. H.

John and Henry passed in to their teacher a written exercise in geometry. On glancing over the two papers, the teacher noticed a striking resemblance in the hand writing. On closer inspection, he found the two diagrams precisely alike each employing the same nine letters, the points being designated by the same letters in both. The teacher at once suspected that Henry had asked and received assistance from John, but refraines from saying anything until he had examined the papers thoroughly. On further inspection of John's paper he discovered two mistakes: then turning to Henry's paper he discovered the same two mistakes in his. The next morning John denied that he had written that exercise for Henry but acknowledged that he had done so once before. When asked by the teacher, however, to pick out his paper from a number of others, the autographs all being concealed, he picked out the paper to which Henry's name was signed. Henry was absent from school that morning, and has not yet returned. What is the probability that John did *not* write Henry's exercise for him?

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

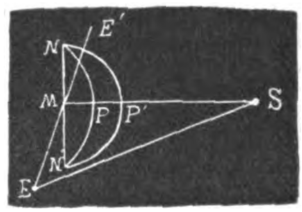
SOLUTIONS OF PROBLEMS.

15. Proposed by SAMUEL HART WRIGHT, M. D., M. A., Ph. D., Penn Yan, Yates County, New York.

Required the illuminated area of the Moon's disc when $\frac{2}{3}$ through its first quarter, or 60° of longitude east of the Sun, the Earth and Moon being at their mean distances.

Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Represent the centers of the sun, earth, and moon, by S, E, and M, respectively; also, represent the *mean* distances of the moon and sun from the earth, respectively, by $EM=d=1$, and $SE=nd=395.40$. Let $\angle MES=\Psi=60°$; $\angle SME'=\phi$; and $MS=d\sqrt{(n^2+1-2n\cos\Psi)},=d\sqrt{(n^2-n+1)}=394.90$; then, by well-known principles of Trigonometry, we deduce



$$\phi=\sin^{-1}\left(\frac{n\sin\Psi}{\sqrt{[(n-1)^2+2n(1-\cos\Psi)]}}\right),=60°7'33''.223.$$

Put $\mathbf{A},=\frac{1}{2}$, the apparent area of the semi-circle $NP'N'$ at a unit's distance, $\mathbf{A}'=$ the apparent area of this semi-circle at a distance $EM=d$; then, since the apparent *diameter* of the moon varies inversely as the distance and the apparent *area* of the moon's disc varies as the square of the distance, we we have $\mathbf{A}'=\mathbf{A}/d^2$. The *phase*, or illuminated area, $NP'N'PN$ at a distance $EM=d$, we represent by $\mathbf{P}$; and the apparent area of the semi-ellipse NPN' at a distance $EM=d$, by $\mathbf{E}$; then, according to obvious principles and deductions already made, we obtain the formula,

$$\mathbf{P}=\mathbf{A}'-\mathbf{E}=\mathbf{A}'(1-\cos\phi)=\mathbf{A}(1-\cos\phi)/d^2,$$

$$=\frac{\mathbf{A}}{d^2}\left[1-\sqrt{\left(\frac{(n-1)^2-n(1-\cos\Psi)[n(1+\cos\Psi)-2]}{(n-1)^2+2n(1-\cos\Psi)}\right)}\right]$$

$=\mathbf{A}(1-.4981517)/d^2=\frac{1}{2}of.5018483=.2509241+$; that is, about *one-forth of the apparent disc* of the moon is then illuminated.

NOTE.—An easy trigonometrical operation gives the formula,

$$\phi=\pi-\left\{\Psi+\tan^{-1}\left[\left(\frac{n-1}{n+1}\right)\cot\tfrac{1}{2}\Psi\right]\right\}.$$

$$=\pi-\left\{\Psi+\tan^{-1}\left[\left(\frac{n-1}{n+1}\right)\sqrt{\left(\frac{1+\cos\Psi}{1-\cos\Psi}\right)}\right]\right\},$$

$$=\pi-[\tfrac{1}{3}\pi+\tan^{-1}(\tfrac{986}{991}\sqrt{3})]=60^\circ\ 7'33''.223.$$

It must be observed that a reversed crescent of the same size is illuminated when the "Waning Moon" is two-thirds through her *last* quarter.

PROBLEMS.

28. Proposed by "IAGO"-(The late DR. JAMES MATTESON, DeKalb Center, Illinois.)

If 9 gentlemen, or 15 ladies, will eat 17 apples in 5 hours, and 15 gentlemen and 15 ladies can eat 47 apples of a similar size in 12 hours, the apples growing uniformly: how many boys will eat up 360 apples in 60 hours, admitting that 120 boys can eat the same number as 18 gentlemen and 26 ladies? *F. P. Matz.*

29. Proposed by ALEXANDER MACFARLANE, M. A., D. Sc., LL.D., Cornell University, Ithaca. New York.

A rectangular room has the four walls, the floor, and the ceiling covered with mirrors: a candle is placed inside the room: find a formula which will express all the images.

EDITORIALS.

Professor Milton L. Comstock, Professor of Mathematics, Knox College, Galesburg, Illinois, says, I have read the MONTHLY from the beginning and I am willing to bear testimony as to its excellence.

In future numbers of the MONTHLY, some valuable contributions on important Mathematical subjects may be looked for from Dr. G. A. Miller, of the University of Michigan, and Dr. W. B. Smith of the Tulane University of Louisiana.

Professor F. P. Matz, of the Department of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland, has just received official notice of his unanimous election as member of the London Mathematical Society, having been proposed for membership, by Professor J. J. Sylvester. Dr. Matz was also elected member of the American Mathematical Society at its meeting in February, having been proposed for election by Professor William W. Johnson, of the United States Naval Academy and endorsed by Professor Simon Newcomb. We congratulate Dr. Matz on these merited recognitions.

Professor J. E. Oliver, Professor of Mathematics in Cornell University, died, on the 27th of March, 1895, after an illness of ten weeks.

On Monday, April 8th, 1 95, Dr. Alexander Macfarlane, was married to Miss Helen Martha Swearingen, of San Antoneio, Texas. Dr. Macfarlane and his wife will make their home in Ithaca, New York. The MONTHLY family wishes them a long and happy life.

BOOKS AND PERIODICALS.

A College Algebra. By J. M. Taylor, A. M., Second edition, 1892. 8vo. cloth, 318 pp. Price, $1.50. Boston and Chicago: Allyn & Bacon.

The first part of this Algebra embraces an outline of those fundamental principles of the science that are usually required for admission to a college or scientific school. The subject of Equivalent Equations and Equivalent Systems of Equations are fully treated. In the Second Part, the author has given a full discussion of the Theory of Limits which is followed by one of its most important appplications, Differentiation. Pages 266-317 are given to the Theory of Equations. The subjects treated in this book are presented with the utmost clearness and simplicity. Teachers of Higher algebra should examine this work. It is neatly printed and well bound in substantial cloth. B.F.F.

Theorems in the Calculus of Enlargement; and *A Method for Calculating Simultaneously all the Roots of an Equation.* By Emory McClintock. Reprinted from the *American Journal of Mathematics*, vol. XVII, Nos. 1 and 2.

Dr. McClintock read his paper on "Theorems in the Calculus of Enlargement" before the American Mathematical Society, August 14, 1894, where it was received with much appreciation. The paper on "A Method for Calculating Simultaneously all the Roots of an Equation" was read before the American Mathematical Society, August 14 and October 27, 1894. This paper presents new results in the main line of analysis which should be immediately incorporated in our text-books. Any one interested in these papers should write to Dr. McClintock, Columbia College, N. Y., for a copy. B.F.F.

The Review of Reviews: An International Illustrated Monthly Magazine. Edited by Albert Shaw. Price, $2.50 per year. Single Number, 25 cents. The Review of Reviews Co. New York City.

In the *Review of Reviews* for April the editor discusses recent political events, especially the doings of the Fifty-third Congress, the appointment of delegates to an international monetary conference, the election of U. S. senators by various state legislatures, the deadlock in Delaware, the constitutional convention in Utah, the arguments before the Supreme Court on the constitutionality of the income tax, the change in the administration of the Post Office Department, and other incidents of the month under review. Persons who can only afford to take one of the leading Literary Magazines of the World and desire to know what is going on all over the world should subscribe for *The Review of Reviews.* B.F.F.

The Cosmopolitan: an Illustrated Monthly Magazine. Edited by

John Brisben Walker and Arthur Sherburne Hardy. Price, $1.50 per year. Single Number, 15 cents.

The following are some of the leading articles in the April Number: The Nymph of the Attitudes, by Mrs. Robert P. Porter: The Late Returning, by Gertrude Hall: English Wood-Notes, by James Lane Allen; The Krakatoa Eruption, by Jean T. VanGestel; and The Story of a Thousand, by Albion W. Tourgee. The description of Krakatoa Eruption, by an eyewitness, is full of thrilling interest from beginning to end. The Cosmopolitan is one of the very best Magazines published in America, and its price is so reasonable that it is easily within the reach of all. You might be charged more than 15 cents for such a Number but could it contain better materal? See our offer in the December No. of the MONTHLY. B.F.F.

***The Mathematical Gazette:* a Terminal Journal for Students and Teachers. Edited by E. M. Langley, M. A., Published by Macmillan & Co., London and New York. No. 4, February, 1895, 4to, pp. 25-36. Price, One Shilling Net. Subscription for 1895, 7s. 6d.**

The February Number contains an excellent paper on "Mathematics for Astronomy and Navagation", by T. Wilson. The paper on "Algebra in Schools", by G. Heppel, offers some good suggestions. The paper suggests that $\sqrt{a}$ should be considered as having one and not two values. Without offering a criticism to this statement, we believe that it is best to consider $\sqrt{a}$ as having always two values, though both values may not be admissable in the same equation. B.F.F.

***The Mathematical Magazine:* a Journal of Elementary and Higher Mathematics. Edited and Published by Artemas Martin, M.A., Ph.D., LL.D., Washington, D. C. Terms: $1.00 in advance for Four Numbers.**

The January, 1895, Number contains the following papers: About Cube Numbers Whose Sum is a Cube, by Dr. Martin: The United States Bond Problem, by Theodore L. DeLand: On the Celebrated Cattle Problem of Archimedes, by A. H. Bell. The solutions of six problems are published and ten new problems are proposed for solution. The appearance and typographical execution of this Number is, as usual, first class. B.F.F.

Nicolai Ivanovich Lobachevsky.

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BIOGRAPHY.

LOBACHEVSKY.

BY DR. GEORGE BRUCE HALSTED.

NICOLAI IVANOVICH LOBACHEVSKY was born November 3, 1793, which, according to the old style calender still used in Russia, is there written October 22, 1793. His father was an architect. This is explicitly stated in the edition of Lobachevsky's works published by the Imperial University of Kasan, so that C. S. Peirce is wrong when in his review of my translation of Vasiliev's Address on Lobachevsky, in the Nation of April 4th, 1895, he criticises Ch. Rumelin for this statement. Rumelin is right. Lobachevsky was born in the town of Makariev about 40 miles above Nizhni-Novgorod on the Volga.

His father died in 1797, and his mother soon after removed to Kasan, where she succeeded in getting her two sons admitted free to the Gymnasium. Lobachevsky entered in November 1802. On February 14th, 1807, after standing his examination, he was admitted as a free student to the University of Kasan, the statues creating which had only been signed November 5th, 1804. Lobachevsky's wildness, disobedience and contempt for orders drew down upon him the severe disapproval of the University authorities; once he was threatened with exclusion from the University, and it was only due to the protection of Bartels, the professor of mathematics, that he was permitted to finish his course.

Toward Bartels Lobachevsky retained to the end of his life the keenest feeling of regard and gratitude.

He studied practical astronomy with Littrow, under whose direction he made observations on the comet of 1811. July 10th, 1811 he received the master's degree, and then began teaching as assistant to Bartels. In 1814 he was made adjunct professor of mathematics, and in 1816 full professor. On May 3rd, 1827, when only 33 years old, he was made Rector, and occupied during 19 years the first place in the University of Kasan. In 1846 he was appointed assistant Curator of the district of Kasan, and went to live in a village which belonged to him, Belovoljskaya Slobodka, sixty versts from Kasan up the Volga, where the forest of nut trees planted by him still remains. Toward the end of his life he became blind, but continued his scientific activity and his complete conviction of the profound importance of his non-Euclidean geometry. His last work "Pangeometrie" was produced after his blindness In 1856 he died.

The researches of Lobachevsky on the systematic interpretation of geometry began before 1823, for in that year he presented to Magnetsky, then Rector, with the idea of having it printed at the charge of the crown, a manual of geometry, written in the "classic" form.

It is agreat pity that this most interesting manuscript is lost. From the detailed judgement of Fus on it we cannot gather that Lobachevsky expressed in this manual any original views on the theory of parallels. This point will be of still greater interest in the future, for I am at present engaged in translating from the Hungarian or Magyar language important documents recently obtained in regard to the two Bolyais, and with them a letter, hitherto unknown and not even yet published, from John Bolyai, the scientific twin of Lobachevsky, in which he announces to his father in 1823 the discovery of the non-Euclidean geometry.

Though Lobachevsky's "Geometric Researches on the Theory of Parallels," published in 1840, of which my English translaton is now in its fourth edition and has been beautifully reproduced in Japan, remains even to-day the simplest introduction to the subject which has ever appeared; yet in it Lobachevsky has not reached that final breadth of view given first in John Bolyas's "Science Absolute of Space," but also attained in Lobachevsky's last work *Pangeometrie*, which name he explicitly uses as expressive of this final view.

There is one point, incomprehensible to his contemporaries, which we can appreciate now as showing the marvellous precision and keenness of Lobachevsky's logic and mathematical perception.

As early as 1834 he mad the distinction in regard to functions, which nearly half a century later Weierstrass and P. du Bois-Reymond forced upon the attention of the mathematical world, namely the distinction between continuity and differentiability. Lobachevsky said, "The function is 'postepennost' [what we now call *continuous*] when the increment in it is diminished to zero together with the increment of the variable x. The function is 'neprerivnost' [what we now call differentiable] when the ratio of these two increments, as they diminish, goes over insensibly into a new function, that will be, consequently, a differential co-efficient." C. S. Peirce says of this, "Who in Russia in 1834 could possibly see anysense in the contention of Lobachevsky that it was one thing for a curved line to be continuous, and quite another for it to have definite tangents? The mathematicians of Western Europe did not become aware of the distinction until nearly 1880, when Weierstrass suggested that a line might be wavy, and these waves carry smaller waves, and those still smaller waves, and so on *ad infinitum*. Down to this day there is but one text-book on the differential Calculus (that of Camille Jordan, in its second edition) which introduces the distinction. All of Lobachevsky's writings are marked by the same high-strung logic."

The solar corona, one of the most remarkable phenomena in nature, was not enough noticed to receive a name until 1851. But it was carefully observed by Lobachevsky at the eclipse of July 8, 1842, and by him minutely described These things are mentioned to show that Lobachevsky was a modern scientists of the very soundest sort, whose only misfortune was to be half-a-century ahead of the world. But as soon as the world reached him it did ample justice to his influence and his memory. In my Bibliography of non-Euclidean Geometry in the American Journal of Mathematics 1878, reproduced by Vastchenko-Zacharchenko in his Introduction to the Elements of Euclid in 1880, and again reprinted in 1886 at the end of the second volume of the collected geometric works of Lobachevsky (Edition of the Imperial University of Kasan), I gave more than a hundred and seventy five works, and in less than ten years, 1887, the number had grown to over three hundred, and now is so enormously great that the task of a new edition of my Bibliography overwhelms me.

The most distinguished men of the generation just passed, Grassmann, Riemann, Helmholtz, Clifford, Cayley; the ablest of living mathematicians, Lie, Klein, Sylvester, Sir R. Ball, Poircare have won some of their choicest honors in the domain of the non-Euclidean geometry. Its day of probation is safely passed, and one might better square the circle and invent perpetual motion than make the slightest objection to the non-Euclidean geometry.

And now in bringing to a close this meagre notice of a wonderful genius, let me say that no attempt has here been made to give an intimate picture of the man himself, because that has already been done to perfection in the magnificent Address of Lobachevsky's worthy successor at Kasan, Professor A. Vasiliev. Of this L. E. Dickson, of the University of Chicago, himself agenius, speaks as follows: "From every one devoted to mathematics or philosophy, or indeed to the highest advance of human thought in any form, this address will call forth the deepest admiration for Lobachevsky, now recognized as one of the greatest intellectual revolutionizers the world has ever had. It will arouse a deeper enthusiasm for scientific achievement and widen the horizon of every reader."

THE "CATTLE PROBLEM." BY ARCHIMEDIES 251 B. C.

By A. H. BELL, Hillsboro, Illinois.

Compute, O stranger! the number of of cattle of Helios, which once grazed on the plains of Sicily, divided according to their color, to wit:

1st White Bulls$=\frac{\text{Black Bulls}}{2}+\frac{\text{Black Bulls}}{3}+$ Yellow Bulls.

2nd Black Bulls$=\frac{1}{4}$ and $\frac{1}{5}$ of the Dappled Bulls+the Yellow.

3rd Dappled Bulls$=\frac{1}{6}$ and $\frac{1}{7}$ of the White Bulls+the Yellow Bulls.

4th The White cows$=\frac{1}{3}$ and $\frac{1}{4}$ of the Black Herd,Bulls and Cows=Herd.

5th The Black cows$=\frac{1}{4}$ and $\frac{1}{5}$ of the Dappled Herd.

6th The Dappled cows$=\frac{1}{5}$ and $\frac{1}{6}$ of the Yellow Herd.

7th The Yellow cows$=\frac{1}{6}$ and $\frac{1}{7}$ of the White Herd.

He who can answer the above is no novice in numbers. Nevertheless he is not yet skilled in wise calculations! but come consider also all the following numerical relations between the Oxen of the Sun.

8th If the White Bulls were combined in one total, with the Black Bulls they would be in a figure equal in depth and breadth and the far stretching plains of Thrinacia would be covered by the figure (square) formed by them.

9th Should the Yellow and Dappled Bulls be collected in one place, they would stand, if they ranged themselves one after another completing the form of an equilateral triangle. If thou discover the solution of this at the same time; if thou grasp it with thy brain; and give correctly all the numbers; O Stranger! go and exult as conqueror; be assured that thou art by all means proved to have abundant of knowledge in this science.—This is translated by T. L. Heath, author of Diophantos, Cambridge, England, 1889.

The first known answer to the Celebrated Cattle Problem by Archimedes 251 B. C. was computed by the Hillsboro, Illinois, Mathematical Club, 1889 to 1893. Edmund Fish, Geo. H. Richards,and A. H. Bell.

The numbers satisfying all of the 9 conditions as given are the very smallest that will meet the requirements and critical tests that are also given. Mathematicians have heretofore obtained the 8th condition which requires the White and Black Bulls to equal a square number, and is 79 450 446 596 004 $=\square$ number; the 9th condition that the Dappled and Yellow Bulls should equal a triangular number is not fullfilled by the corresponding number, 51, 285 802 909 803, which is designated by B. We seek a square multiplier which call x^2 let $Bx^2=\frac{n\ (n+1)}{2}=$ the expression for a triangular number which gives $8Bx^2+1=(2n+1)^2=y^2$ and we at once get $8B=\frac{1\ (y^2-1)}{x}$. The square root of $8B$ by continued fractions will give x, and then we have

$x^2 =$ 34 555 906 354 559 370 506 303 802 963 617+68 829 periods of 3's+252 058 980 100.

White Bulls 1 596 510 804 671 144 531 435 526 194 370+68 834 periods of 3's+385 150 341 800.

Black Bulls 1 148 971 387 728 289 999 712 359 821 824+68 834 periods of 3's+899 825 178 600

Dappled Bulls 1 133 192 754 438 638 077 119 555 879 202+68 834 periods of 3's+921175 894 000

Yellow Bulls 639 034 648 230 902 865 008 559 676 183+68 834 periods of 3's+635 296 026 300

White Cows 1 109 829 892 373 319 039 723 960 215 824+68 834 periods of 3's+914 059 564 000

Black Cows 753 594 142 054 542 639 814 429 119 589+68 834 periods of 3's+238 562 645 400

Dappled Cows 541 460 894 571 456 678 023 619 942 106+68 834 periods of 3's+608 963 318 000

Yellow Cows 837 676 882 418 524 438 692 221 984 107+68 834 periods of 3's+116 422 113 700

Total 7 760 271 406 486 818 269 530 232 833 209+68 834 periods of 3's+719 455 081 800

W. and B. Bulls 2 745 482 192 399 434 531 147 886 016 194+68 834 periods of 3's+284 975 520 400

Root of above 1 656 949 665 133 506 668+34 414 periods of 3's+357 460 163 020

D. & Y. = △ = 1 772 227 402 669 540 942 128 115 555 385+68 834 periods of 3's+556 471 920 300

Root of 8△+1 3 765 344 502 347 205 884+34 414 periods of 3's+363 134 961 201

These enormous numbers using 206545 figures will make numbers one-half mile long. In the computations to this problem difficulties are encountered at every step, wonderfull discoveries in the properties of vast numbers are disclosed at every turn. A new summation of continued fractions with many novel ways used to obtain the exact figures shown can be had of A. H. Bell, Hillsboro, Illinois.

REMARKS ON SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Professor of Mathematics University of Michigan, Ann Arbor, Michigan.

Introductory. "The mathematics of the twenty-first century may be very different from our own; perhaps the school boy will begin algebra with the theory of substitution groups as he might now but for inherited habits."*

These words imply two important features of substitution groups, viz., their extensive application and their rudimentary character, for a beginners' algebra must in all times possess these features. To these we may perhaps add a third, viz., their usefulness for unification; for we can scarcely expect that much will be added to the beginners' algebra except what tends to show the unity of subjects which otherwise appear distinct and thus to simplify the acquisition of a knowledge of them. The last one of these features is especially prominent in substitution groups and on this account it seems probable that they will increase in favor with the progress of thought towards abstraction and generalization.

The theory of substitution groups does not concern itself with metrical properties nor is it a part of the science of quantity in the ordinary sense. It chiefly investigates correspondencies by means of which we may apply results with respect to one subject to another without going through the work of investigating each seperately. It tends therefore to save thought and thus becomes a new instrument in the hands of the student by means of which he can work not only faster but also with less exertion than he could without employing it. With these general statements in regard to the scope and features of our subject we proceed to some special consideration of its nature and a few important applications. In doing this we shall not presuppose any knowledge of the subject on the part of the reader.

If we take the four numbers $1, -1, \sqrt{-1}, -\sqrt{-1}$ and multiply any one by itself or by another one of them we obtain no new numbers; thus

$$-\sqrt{-1}\times-\sqrt{-1}=-1,\quad -\sqrt{-1}\times-1=\sqrt{-1}$$
$$-\sqrt{-1}\times\sqrt{-1}=1,\quad -1\times-1=1, \text{ etc.}$$

We therefore say that these four numbers form a *group* with respect to multiplication. If we take 1 and operate upon it by itself we obtain no new number. 1 therefore forms a *subgroup*† of the given group. If we take -1 in a similar way we obtain the subgroup $1, -1$ since $(-1)^2=1, (-1)^3=-1$, etc. By taking either $\sqrt{-1}$ or $-\sqrt{-1}$ in this way we will obtain the entire group, since $(\sqrt{-1})^2=-1, (\sqrt{-1})^3=-\sqrt{-1}, (\sqrt{-1})^4=1, (\sqrt{-1})^5=\sqrt{-1}$, etc. We say therefore that the above group is *generated* by either $\sqrt{-1}$ or $-\sqrt{-1}$ and that $1, -1$ generate the subgroups 1 and $(1, -1)$ respectively.

*From Professor Simon Newcomb's address delivered before the New York Mathematical Society at its annual meeting, December 28, 1893, and published in the *Bulletin* of this society, vol. III., No. 4.

† The terms group and subgroup are only relative. If a subgroup is considered by itself it is called a group and a group may in turn be a subgroup of a larger group.

We may now consider the corresponding *substitution* group

1, *ac.bd*, *abcd*, *adcb*.

Where *ac.bd* means that *c* is substituted for *a* and *a* for *c*, then *d* for *b* and *b* for *d*; *abcd* means that *b* is substituted for *a*, *c* for *b*, *d* for *c* and *a* for *d*. 1 means that all the letters are replaced by themselves, or, what amounts to the same, that the letters all remain unchanged.

For example, if we operate upon $a+2b+3c+4d$ with *ac.bd* we obtain $c+2d+3a+4b$ and if we operate upon the same expression with *abcd* and *adcb* we obtain $b+2c+3d+4a$, $d+2a+3b+4c$ respectively. From these definitions it follows directly that 1, *ac.bd* generate the subgroups 1 and (1,*ac.bd*) respectively, for if we perform the operation indicated by these symbols any number of times and in any order we obtain a result which is equivalent to the operation indicated by one of the symbols in these subgroups. This is the necessary and sufficient condition that a given aggregate of symbols may be a *group* or subgroup. Since $abcd^2$* $=ac.bd$, $abcd^3=adcb$, $abcd^4=1$, $abcd^5=abcd$, etc., it follows that the above group is generated by either of the substitutions *abcd* or *adcb*.

In the study of substitution groups it is very important to be familiar with quite a number of such groups. We therefore shall give all the groups of the different numbers of letters not exceeding four. It is evident that the only substitution group of two letters is

1, *ab*.

The following are the only two that involve three letters:

1	*abc*	*ab*
	acb	*ac*
		bc

1	*abc*
	acb

The following seven are those that involve four letters:

1	*abc*	*abcd*	*ac*	*ab.cd*
	acb	*adcb*	*ab*	*ac.bd*
	abd	*acbd*	*ad*	*ad.bc*
	adb	*adbc*	*bc*	
	acd	*abdc*	*bd*	
	adc	*acdb*	*cd*	
	bcd			
	bdc			

1	*abc*	*ab.cd*
	acb	*ac.bd*
	abd	*ad.bc*
	adb	
	acd	
	adc	
	bdc	
	bdc	

1	*ab.cd*	*abcd*	*ac*
	ac.bd	*adcb*	*bd*
	ad.bc		

1	*ab.cd*
	ac.bd
	ad.bc

1	*abcd*	*ac.bd*
	adcb	

1	*ac*	*ac.bd*
	bd	

1	*ac.bd*

* This means that the operation indicated by *abcd* is to be performed twice in succession. In general $abcd^x = abcd.abcd.....abcd$, the operation *abcd* being performed *x* times in succession. If we operate once upon $a+2b+3c+4d$ with *abcd* we obtain $b+2c+3d+4a$, on operating again we obtain $c+2d+3a+4b$; the last result would have been obtained by operating with *ac.bd* upon the first expression.

The largest group of any given number of letters contains all the substitutions which correspond to the permutations of these letters. The number of substitutions in this group is therefore $n!$ and all the other groups of the same number of letters are subgroups of it. The second group, with respect to size contains all the substitutions which correspond to an even number of interchanges of letters, the number of substitutions in this group is $n!\div 2$.

[To be continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton), Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the April Number.]

PROPOSITION XVII. *In the hypothesis of acute angle, we can find a perpendicular and an oblique to the same straight which never meet.*

More literally: *If the straight AH stands* (fig. 15.) *at right angles to any one certain straight AB however small: I say that in the hypothesis of acute angle it cannot hold good, that every straight BD, making with AB any acute angle you choose, toward the parts of this AH, will at length meet this AH produced at a finite, or terminated distance.*

PROOF. Join HB. The angle ABH will be acute (Eu. I. 17), because of the right angle at the point A. Now (Eu. I. 23) a certain HD can be drawn toward the parts of the point B, which not cutting the angle AHB makes with this HB an acute angle equal to this acute angle ABH. Then from the point B is let fall to HD the perpendicular BD, which falls toward the parts of the aforesaid acute angle at the point H. Since therefore the side HB is opposite in the triangle HDB to the right angle at D, and likewise in the triangle BAH to the right angle at A; and again in those two triangles equal angles are adjacent to this side HB, which are in the first triangle indeed the angle BHD, and in the latter the angle HBA; also (Eu. I. 26) the remaining angle HBD in the former triangle will be equal to the remaining angle BHA in the latter triangle. Wherefore the entire angle DBA will be equal to the entire angle AHD. Now however, neither of the aforesaid equal angles will be obtuse, lest we meet (from the preceding proposition) in one case the now rejected hypothesis of obtuse angle. Nor will

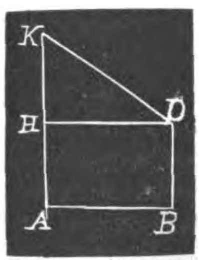


Fig. 15.

either be right, lest we meet (from the same preceding) in one case accordingly the hypothesis of right angle, which (P.V.) will leave no place for the hypothesis of acute angle. Therefore each one of those angles will be acute. This being the case; that the straight *BD* produced cannot meet in a certain point *K* this *AH* produced toward the same parts, is demonstrated thus; because in the triangle *KDH*, besides the right angle at *D*, is present the obtuse angle at *H*, since the angle *AHD* in the aforesaid hypothesis of acute angle is proved acute. But this is absurd, against Eu. I. 17. Therefore it cannot hold good in this hypothesis, that any *BD*, making with one straight *AB* as small as you choose, any acute angle towards the parts of this *AH*, will at length at a finite, or terminated distance, meet this *AH* produced. Quad erat demonstrandum.

The same otherwise and more easily. Two perpendiculars *AK, BM* stand on one certain small at will straight *AB* (fig. 16). From any point *M* of this *BM* let fall the perendicular *MH*, and join *BH*. It follows that the angle *BHM* will be acute. In the hypothesis of acute angle, the angle *BMH* is also (from the preceding proposition) acute. Therefore the perpendicular *BDX*, let fall from the point *B* to this *HM*, will cut (by Eu. I. 17) this *HM* in some intermediate point *D*. Therefore the angle *XBA* will be acute. But it follows (from the same Eu. I. 17) that those two straights *AHK, BDX* howsoever produced cannot mutually meet (anyhow at a finite, or terminated distance) on account of the right angles at the points *H* and *D*. Therefore in the hypothesis of acute angle it cannot hold good, that any *BD*, making with one however small straight *AB* any acute angle toward the parts of this *AH*, perpendicular to this same *AB*, will at length meet (at a finite, or terminated distance) this *AH* produced. Quad erat propositum.

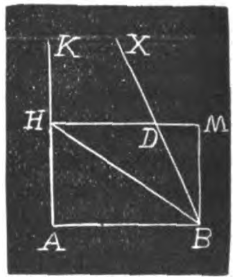


Fig. 16.

Scholion i. And this is, what I promised in the Scholia after proposition XIII, that the hypothesis of acute angle (which alone is able now to stand against that general Postutatum of Euclid) will certainly be destroyed by the sole admission of a general meeting of two straights toward those parts, toward which any straight, as small as you choose, meeting them, makes two internal angles less than two right angles; and so indeed, even if either of those angles is to be supposed right.

Scholion ii. But again in a better place, after proposition XXV, I will show that the hypothesis of acute angle will be equally destroyed, provided that any one acute angle as small as you choose can be designated, under which if any straight meets another, this produced must (at a finite, or terminated distance) finally meet any perpendicular erected upon this meeting straight at whatever finite distance.

Proposition xviii. *From any triangle ABC, of which* (fig. 17.) *the angle at the point B is inscribed in any semicircle, of which the diameter is AC, is established the hypothesis of right angle, or obtuse angle, or acute angle, according as indeed the angle at the point B was right, or obtuse, or acute.*

PROOF. From the center D join DB. The angles at the base AB will be (Eu. I. 5) equal, and likewise at the base BC, in the triangles ADB, CDB. Wherefore in the triangle ABC the two angles at the base AC will be together equal to the whole angle ABC. Therefore the three angles of the triangle ABC will be together equal to, or greater, or less than two right angles according as the angle at the point B was right, or obtuse, or acute. Therefore from any triangle ABC, of which the angle at the point B is inscribed in any semicircle, whose diameter is AC, is established (P.XV) the hypothesis of right angle, or obtuse angle, or acute angle, according as indeed the angle at the point B is right, or obtuse, or acute.

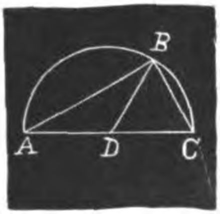


Fig. 17.

Quod erat demonstrandum.

[To be continued.]

AN ATTEMPT TO DEMONSTRATE THE 11th AXIOM OF PLAYFAIR'S EUCLID.

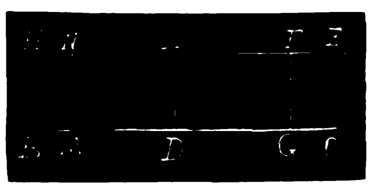
By WARREN HOLDEN, Professor of Mathematics. Girard College, Philadelphia, Pennsylvania.

"*Through a given point one line, and only one, can be drawn parallel to a given line.*"

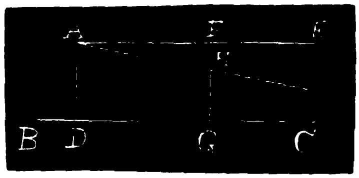
1st. Two lines perpendicular to a third never intersect, how far soever they be produced. *Halsted's Lobatschewsky's Geometry.* Page 12. Art. 4.

2d. Parallel straight lines are such as are in the same plane, and which, being produced ever so far both ways, do not meet.

3d. Paralles are everywhere equidistant. From A draw AD perpendicular to BC, and through A draw HAE perpendicular to AD. AE and DC being perpendicular to the same line AD are parallel (1st and 2d). From *any* point F let fall the perpendicular FG upon BC. Lay off DM equal to DG, an erect the perpendicular MN. Fold over the part of the figure to the right of AD upon AD as an axis until it falls upon the part to the left. Since A, D, G and M are right angles, and since DM equals DG by construction, AF must fall upon AN, and GF upon MN. The point F is found upon AN and MN, at their intersection N. Therefore GF equals MN. Since F is *any* point, the parallels are everywhere equidistant.



4th. Through a given point one line, and only one, can be drawn parallel to a given line. Draw AD perpendicular to BC, and through A draw AE perpendicular to AD. Then (1st and 2d) AE and DC are parallel; and the perpendicular FG equals AD, by (3d). Now suppose another line AH parallel to BC. Then HG equals AD or its equal FG. When HG equals FG, AH and AF coincide. Therefore, through a given point one line, and only one, can be drawn parallel to a given line.



The above demonstration may be made without the use of the word parallel. Thus: Through a given point one line, and only one, can be drawn equidistant from a given line.

With the figure drawn as in No. 3, begin the demonstration with the words: From *any* point F let fall the perpendicular &c., to prove the lines equidistant. Then with the same figure as in No. 4, and substituting the word equidistant for parallel, we have the demonstration.

CRADLE-ROCKING BY ELLEPTIC FUNCTIONS.

By F. P. MATZ, M. Sc., Ph. D., New Windsor, Maryland.

After adopting the *gravitation-unit* of force, the equation of motion of the pendulum may be written $(h^2+k^2)(W/g)(d^2\theta/dt^2)=-Wh\sin\theta$....(1). Briefly making $(h+k^2/h)=l$ and $g/l=n^2$, we obtain from (1)

$\frac{1}{2}(d\theta/dt)^2=n^2(\text{vers }\alpha-\text{vers}\theta)$....(2), in which, according to Sir William Thomson (Lord Kelvin), n is the *angular speed* of the pendulum, Divide *semicircularly* the pendulum-bob, turn downward the convex sides of these divisions centrally joined by a rectilinear axis of inappreciable length, and let the pendulum-rod bisect this rectilinear axis. In the position specified, these divisions constitute the *rockers* of an old-fashioned cradle; and this cradle we regard as placed upon a perfectly rough horizontal plane. Detaching the pendulum rod from the point of suspension, we have to consider *the rocking*, or the rolling oscillations on a horizontal plane, of a material body resting on a semicircular base. Let r=the radius of the equal semicircular rockers. Consider the *line* joining the points of tangency of the rockers with the horizontal plane, as the instantaneous axis of rotation; then, after obvious transformations, (2) becomes $\frac{1}{2}(r^2-2hr\cos\theta+h^2+k^2)(d\theta/dt)^2=gh(\text{vers}\alpha-\text{vers}\theta)$....(3).

$$\therefore \left(\frac{dt}{d\theta}\right)^2 = \frac{[(r-h)^2+k^2]\cos^2\frac{1}{2}\theta+[(r+h)^2+k^2]\sin^2\frac{1}{2}\theta}{4gh(\sin^2\frac{1}{2}\alpha-\sin^2\frac{1}{2}\theta)},$$

$$= \frac{[(r-h)^2+k^2]\cos^2\frac{1}{2}\theta+[(r+h)^2+k^2]\sin^2\frac{1}{2}\theta}{4gh[\sin^2\frac{1}{2}\alpha-(1-\sin^2\frac{1}{2}\alpha)\tan^2\frac{1}{2}\theta]},$$

$$= \frac{[(r-h)^2+k^2]+[(r+h)^2+k^2]\tan^2\frac{1}{2}\theta}{4gh\cos^2\frac{1}{2}\alpha(\tan^2\frac{1}{2}\alpha-\tan^2\frac{1}{2}\theta)} \ldots\ldots (4).$$

In order to transform (4), put $\tan\frac{1}{2}\theta=\tan\frac{1}{2}\alpha\cos\phi$; then differentiating,

$$\text{etc.,}\left(\frac{d\theta}{d\phi}\right)^2 = 4\left(\frac{\tan\frac{1}{2}\alpha\sin\phi}{1+\tan^2\frac{1}{2}\alpha\cos^2\phi}\right)^2, = 4\left(\frac{\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha\sin\phi}{1-\sin^2\frac{1}{2}\alpha\sin^2\phi}\right)^2 \ldots\ldots (a).$$

$$\therefore \left(\frac{dt}{d\phi}\right)^2 = \frac{[(r-h)^2+k^2]\cos^2\frac{1}{2}\alpha+(1-\sin^2\phi)[(r+h)^2+k^2]\sin^2\frac{1}{2}\alpha}{gh(1-\sin^2\frac{1}{2}\alpha\sin^2\phi)^2} \ldots\ldots (5).$$

$$\text{Put}\quad \varkappa^2 = \frac{[(r+h)^2+k^2]\sin^2\frac{1}{2}\alpha}{[(r+h)^2+k^2]\sin^2\frac{1}{2}\alpha+[(r-h)^2+k^2]\cos^2\frac{1}{2}\alpha} \ldots\ldots (b),$$

$$\text{and}\quad \varkappa'^2 = \frac{[(r-h)^2+k^2]\cos^2\frac{1}{2}\alpha}{[(r+h)^2+k^2]\sin^2\frac{1}{2}\alpha+[(r-h)^2+k^2]\cos^2\frac{1}{2}\alpha} \ldots\ldots (c);$$

then will $\varkappa^2+\varkappa'^2=1$. Make $n=\sqrt{(g/r)}$, and represent the denominators of (b) and (c) by M; then (5) may be written

$$ndt = \sqrt{\left(\frac{M}{hr}\right)}\left[\frac{(1-\varkappa^2\sin^2\phi)d\phi}{(1-\sin^2\frac{1}{2}\alpha\sin^2\phi)\sqrt{(1-\varkappa^2\sin^2\phi)}}\right] \ldots\ldots (6).$$

According to the Jacobian system of notation as modified by Gudermann (*Theorie der Modular Functionen*), we have $\phi=\operatorname{am}U$, and $\sin\frac{1}{2}\alpha=\varkappa\operatorname{sn}A$.

Since $\operatorname{dn}^2A+\varkappa^2\operatorname{sn}^2A=1$, we obtain $\operatorname{dn}^2A=1-\varkappa^2\operatorname{sn}^2A=1-\varkappa^2(\sin^2\frac{1}{2}\alpha/\varkappa^2)=\cos^2\frac{1}{2}\alpha$; also,

$$\operatorname{sn}A = \sqrt{\left(\frac{[(r+h)^2+k^2]\sin^2\frac{1}{2}\alpha+[(r-h)^2+k^2]\cos^2\frac{1}{2}\alpha}{[(r+h)^2+k^2]}\right)} \ldots\ldots (d),$$

$$\text{and}\ \operatorname{cn}A=\sqrt{(1-\operatorname{sn}^2A)}, = \frac{2\cos\frac{1}{2}\alpha\sqrt{(hr)}}{\sqrt{[(r+h)^2+k^2]}} \ldots\ldots (e).$$

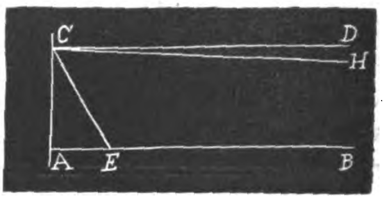
$$\therefore\ ndt = 2\left(\frac{\operatorname{sn}A\operatorname{dn}A}{\operatorname{cn}A}\right)\left[1-\frac{\varkappa^2\operatorname{cn}^2A\operatorname{sn}^2U}{1-\varkappa^2\operatorname{sn}^2A\operatorname{sn}^3U}\right]dU \ldots\ldots (7),$$

and $nt=2[(\operatorname{sn}A\operatorname{dn}A/\operatorname{cn}A)\,U-\Pi(U,A)] \ldots\ldots (8)$, while $\tan\frac{1}{2}\theta=\tan\frac{1}{2}\alpha\operatorname{cn}U$.

II. LOGICAL DEDUCTIONS FROM THE HYPOTHESIS THAT THE ANGLE SUM IS LESS THAN TWO RIGHT ANGLES.

By JOHN N. LYLE, Ph. D., Professor of Natural Science in Westminster College, Fulton, Missouri.

Erect the perpendiculars AB and CD to the straight line AC at the points A and C, On AB lay off $AE=AC$ and draw a straight line from C to E.



By construction the triangle ACE is isosceles. Hence, the angle $AEC=ACE$.

By hypothesis the angle sum of every triangle and, hence, of ACE is supposed to be less than two right angles. In accordance with this assumption let the angle sum of the triangle ACE be equal to two right angles$-a$. Construct $DCH=a$. Then $CAE+ACE+AEC=CAE+ACH$.

Subtract $CAE+ACE$ from both members. Then $AEC=ECH$.

But AEC and ECH are alternate angles. Hence, CH is parallel to AB in the Euclidian sense, that is, it will not meet AB however far both lines may be produced. Euclid. Book I. Proposition XXVII.

Therefore, when the angle sum is assumed in any instance to be equal to $CAE+ACH$, the line CH can not consistently with that hypothesis meet AB.

If, however, the angle sum is assumed to be greater than $CAE+ACH$, it is consistent with this hypothesis to suppose that the line CH may meet AB. For if we make this supposition a triangle will be formed whose angle sum is greater than $CAE+ACH$.

Further, it is inconsistant with the hypothesis to suppose that the line CH can not meet AB. For to deny that CH can meet AB is to deny that a triangle whose angle sum is greater than $CAE+ACH$ can be formed, which is to deny the hypothesis.

But the deduction that CH may meet AB contradicts the conclusion that CH can not meet AB. Therefore, the hypothesis that the angle sum may be greater than $CAE+ACH$ is inconsistant with the hypothesis that the angle sum in any instance is equal to $CAE+ACH$.

That is, the hypothesis that the angle sum is a variable less than two right angles approaching two right angles as a limit is contradictory and hence absurd.

If the hypothesis that the angle sum is less than two right angles is false, sound science requires that the logical deductions from the hypothesis should likewise be false

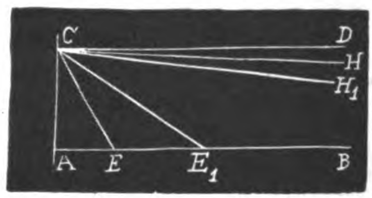
One deduction is that the lines AB and CH making angles with AC whose sum is less than two right angles do not meet. This contradicts Euclid's axiom 12.

Another deduction is that the alternate angles AEC and ECD are not equal although the perpendiculars AB and CD to AC are parallel to each other and can not meet.

Another deduction is that through the same point two straight lines may be drawn parallel to the same straight line. This contradicts the statement known as Playfair's axiom.

Still another of these deductions is that if one side of a triangle be produced the exterior angle is greater than the sum of the two interior and opposite angles.

Lay off $EE_1 = CE$ and draw a straight line from C to E_1.



By hypothesis $ECE_1 + EE_1C + CEE_1 < 2$ right angles. But $AEC + CEE_1 = 2$ right angles. Hence $ECE_1 + EE_1C + CEE_1 < AEC + CEE_1$, and $ECE_1 + EE_1C < AEC$.

Add $EAC + ACE$ to both members of this inequality. Then $E_1AC + ACE_1 + AE_1C < EAC + ACE + AEC$. That is, the angle sum of ACE_1 is less than that of ACE.

Let the angle sum of $ACE_1 = 2$ right angles$-b$. But the angle sum of $ACE = 2$ right angles$-a$.

Hence, $b > a$.

Construct $DCH_1 = b$.

Then $CAE_1 + ACE_1 + AE_1C = CAE_1 + ACH_1$, in which $ACH_1 < ACH$.

Subtract $CAE_1 + ACE_1$ from both members. Then $AE_1C = E_1CH_1$.

But AE_1C and E_1CH_1 are alternate angles. Hence, CH_1 is parallel to AB in the Euclidian sense, that is, it will not meet AB however far both lines may be produced. Euclid. Book I. Proposition XXVII.

Therefore, when the angle sum is assumed in any instance to be equal to $CAE_1 + ACH_1$, the line CH_1 can not consistently with that hypothesis meet AB.

The angle sum of ACE is assumed to be $CAE + ACH$, that is, greater than $CAE_1 + ACH_1$.

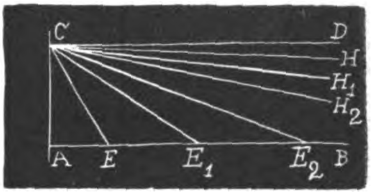
If greater, the line CH_1 may consistently with the hypothesis meet AB.

But the deduction that CH_1 may meet AB contradicts the conclusion already reached that CH_1 can not meet AB.

Therefore, the hypothesis that the angle sum may be greater than $CAE_1 + ACH_1$ is inconsistent with the hypothesis that the angle sum in any instance is equal to $CAE_1 + ACH_1$. That is, the hypothesis that the angle sum is a variable less than two right angles approaching two right angles as a limit in value is a contradiction and is therefore false.

Let us proceed with our investigation. Construct the successive isosceles triangles CE_1E_2, CE_2E_3,&c.

Let us consider the series of triangles AE_2C, AE_2C,&c. From the hypothesis that the angle sum is less than two right angles, the conclusion is reached by a process used above in this article that the angle sum of each triangle is less than that of the preceeding triangles in the series.



Draw the lines CH_2, CH_3,&c., making $CAB+ACH_2$ equal to the angle sum of the triangle AE_2C, and $CAB+ACH_3$ equal to the angle sum of the triangle AE_3C,&c. It then follows that $DCH_1 < DCH_2$ and $DCH_2 < DCH_3$,&c.

These results contradict Euclid. They seem also to be inconsistent with each other, for they apparently teach that the lines CH, CH_1, CH_2, CH_3,&c., both do and do not meet AB. Furthermore the inconsistency is interminable inasmuch as the series DCH, DCH_1, DCH_2, DCH_3,&c., is non-terminating.

Lobatschewsky in enunciating his doctrine of "Imaginary Geomotry" expressly calls his triangle "rectilineal." The logical, geometrical and metaphysical difficulties that follow the denial of the Euclidian axiom 12 and the Euclidian angle sum are so great, however, that non Euclidian writers are now maintaining that Lobatschewsky's triangle can not be drawn in a Euclidian plane and that it is not in fact rectilineal. Since this homeless, outcast triangle is unable to find a "local habitation" in the space of the Alexandrian geometer, the non-Euclidians have excogitated a space especially to contain it called by them "pseudo spherical." Helmholtz in his Lecture "On the origin and significance of geometrical axioms" refers to a "pseudo spherical surface" as "saddle-shaped." He says that the Italian Mathematician E. Beltrami investigated its properties and gave it the name pseudo spherical. Later on in his Lecture he dexterously passes from the phrase—"pseudo spherical surface" to pseudo spherical space." This performance is plainly pseudological. Surface manifestly is not identical with space. Surfaces may be located in space but should not be confounded with space. Beltrami contributes to modern geometrical literature the expression "pseudospherical surface." Helmholtz treats it as identical with "pseudo spherical space" by pseudo logical reasoning and pseudo philosophical speculation.

ARITHMETIC.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

44. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

A, B, and C together bought a ship. A paid for the a/bth, $=\frac{2}{9}$th, part of the ship. B paid for the m/nth, $=\frac{3}{5}$th, part of the ship. C paid $\$M$, $=\$2000$. How many dollars did A, and B, pay?

I. Solution by E. R. ROBBINS, Master of Mathematics in Lawrenceville Schools, Lawrenceville, N. J., COOPER D. SCHMITT, Professor of Mathematics, University of Tennessee, Knoxville, Tennessee, and the PROPOSER.

Since C paid for the $[1-(a/b+m/n)]$th *part* of the ship, the *amount* A paid must be

$$A=\left(\frac{an}{b(n-m)-an}\right)\text{of } \$M, =\left(\frac{1}{b/a(1-m/n)-1}\right) \text{ of } \$M, =\$2500;$$

and, consequently, the amount B paid must be

$$B=\left(\frac{bm}{b(n-m)-an}\right)\text{of } \$M, =\left(\frac{1}{n/m(1-a/b)-1}\right) \text{ of } \$M, =\$6750.$$

Note.—The generalized expression for the cost of the ship becomes

$$S=\left(\frac{bn}{b(n-m)-an}\right)\text{of } \$M, =\left(\frac{1}{(1-m/n)-a/b}\right) \text{ of } \$M, =\$11250.$$

II. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia, and W. I. TAYLOR, Instructor in Mathematics, Berea, Ohio.

$\frac{2}{9}+\frac{3}{5}=\frac{10}{45}+\frac{27}{45}=\frac{37}{45}$; $\frac{45}{45}-\frac{37}{45}=\frac{8}{45}$, C's share. $\frac{8}{45}=\$2000$, $\frac{1}{45}=\frac{1}{8}$ of $\$2000=\$250$.
$\frac{10}{45}=10\times\$250=\$2500$, what A, pays. $\frac{27}{45}=27\times\$250=\$6750$, what B, pays.

III. Solution by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

C pays for the $1-\left(\frac{a}{b}+\frac{m}{n}\right)$ part of the ship; hence, the price of the ship is $M\div\left[1-\left(\frac{a}{b}+\frac{m}{n}\right)\right]$. A's share $=\frac{a}{b}\cdot\frac{M}{1-(a/b+m/n)}=\$2500$.

$$B\text{'s}=\frac{m}{n}\cdot\frac{M}{1-(a/b+m/n)}=\$6750.$$

Also solved by P. S. Berg.

45. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

In running a mile, A can give B $a=20$ yards; B can give C $b=88$ yards. How may yards can A give C?

I. Solution by P. S BERG, Apple Creek, Ohio.

1 mi. $-\frac{1}{88}$ mi. $=\frac{87}{88}$ mi., distance B runs while A runs a mile.

1 mi.$-\frac{1}{20}$ mi. $=\frac{19}{20}$ mi., distance C runs while B runs a mile. $\frac{87}{88}\times\frac{19}{20}=\frac{1653}{1760}$mi. distance C runs while A runs a mile.

Hence 1 mi $-\frac{1653}{1760}$ mi $=\frac{107}{1760}$ mi $=107$ yards, the distance A can give C.

II. Solution by E. L. SHERWOOD, Houston, Mississippi.

A runs a mile while B runs 1740 yards. B runs a mile while C runs 1672 yards or C runs $\frac{1672}{1760}$ of B's distance. So A runs 1760 yards; B, 1740 yards, and C, $\frac{1672}{1760}$ of 1740 yards or 1653 yards.

Whence A can allow C 1760 yards$-$1653 yards or 107 yards.

This problem was also solved by *Cooper D. Schmitt, W. I. Taylor, G. B. M. Zerr, J. F. W. Scheffer E. R. Robbins*, and the *Proposer*.

PROBLEMS.

50. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

If A walk to the city and ride back, he will require $m=5\frac{1}{4}$ hours; but if he walk both ways, he will require $n=7$ hours. How many hours will he require to ride both ways?

51. Proposed by F. P. MATZ. M. Sc., Ph. D., Professor of Mathematics and Astronomy, in New Windsor College, New Windsor, Maryland.

A banker, in discounting a note due in $m=4$ months at $r=3\%$ per annum, charges $C=\$12\frac{1}{4}$ more than the true discount. What is the face of the note discounted?

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

41. Proposed by A. H. BELL, Hillsboro, Illinois.

In a right-angled triangle there are given, the bisectors of the acute angles. Required the triangle.

I. Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland, and D. G. DORRANCE, Jr., Camden, N. Y.

Represent the half-angles by α and $(45-\alpha)$; then easily is deduced

$$\tan 2\alpha = n\cos(45-\alpha)/m\cos\alpha \ldots.(1).$$

$$\therefore \quad \tan^3\alpha+\tan^2\alpha+[(2m\sqrt{2}-n)/n]\tan\alpha-n=0\ldots.(2);$$

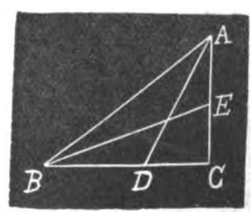
that is, $(\tan\alpha-Q_1)(\tan\alpha-Q_2)(\tan\alpha-Q_3)=0\ldots.(3)$.

Hence three sets of values of the sides of the required right triangle are possible. *Numeralizing* m and n in (2), we deduce Q_1, Q_2, and Q_3 from (3); then α is known. Consequently the three sides, $b=m\cos\alpha$, $p=n\cos(45-\alpha)$, and $h=m\cos\alpha\sec 2\alpha$, are known.

II. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let $BC=x$, $AC=nx$, $AB=mx$, $AD=a$, $BE=b$.
Then $m^2-n^2=1\ldots.(1)$.

$$ma+na=2mnx\cos\tfrac{1}{2}A=\frac{2mn^2x^2}{a}\ldots.(2),$$



$$b+mb=2mx\cos\tfrac{1}{2}B=\frac{2mx^2}{b}\ldots.(3).$$

$$(2)\div(3)\text{ gives }\frac{(m+n)a^2}{(m+1)b^2}=n^2\ldots.(4).$$

Eliminating n between (1) and (4),

$$m^6+2m^5-\left(1+\frac{2a^2}{b^2}\right)m^4-\left(4+\frac{2a^2}{b^2}\right)m^3-\left(1-\frac{2a^2}{b^2}\right)m^2+2\left(1+\frac{a^2}{b^2}\right)m+\frac{a^4}{b^4}+1=0.$$

Let $\frac{a}{b}=u$. Then $m^6+2m^5-(1+2u^2)m^4-(4+2u^2)m^3-(1-2u^2)m^2+2(1+u^2)m+u^4+1=0$.

To give a more complete solution of this equation might be interesting but well-nigh impossible unless we use numerical results. Such a solution, however, is as unsatisfactory as the problem itself.

Let $u^2=\frac{128}{81}$. Then $3m=5$ or $m=\frac{5}{3}$, $n=\frac{4}{3}$, $x=3$.

$\therefore$ $mx=5$, $nx=4$, $x=3$. Let $u^2=4$. Then $m=1.332$, $n=.8799$, $x=.936-$. $\therefore$ $mx=1.246$, $nx=.8235$, $x=.936$, when $a=2$, $b=1$.

Let $a=40$, $b=50$, $u^2=\frac{16}{25}$, Then $m=1.2532$, $n=.7553$, $x=47.4072+$.

$\therefore$ $mx=59.4107+$, $nx=35.8067+$, $x=47.4072+$.

Let $a=b=c$, then $u^2=1$, $m=\sqrt{2}$, $n=1$, $x=\frac{c}{2}\sqrt{(2+\sqrt{2})}$.

$\therefore$ $mx=\frac{c}{\sqrt{2}}\sqrt{(2+\sqrt{2})}$, $nx=x=\frac{c}{2}\sqrt{(2+\sqrt{2})}$.

III. **Solution by B F. BURLESON, Oneida Castle, New York**

Let ABC be the triangle, right angled at C. Put $AD=a=40$, and $BE=b=50$, the lines bisecting the acute angles A and B. Put $x=AB$, $y=AC$, and $z=BC$. Put $\phi+\theta=$the$\angle CAD$ and $\phi-\theta=$the $\angle CBE$. We have, by Trigonometry,

$$x=b\cos(\phi-\theta),\ldots.(1),$$
$$y=a\cos(\phi+\theta),\ldots.(2),$$
$$y=z\tan(2\phi-2\theta)\ldots.(3),$$

Eliminating from (1), (2), and (3), we obtain by development

$(b+b\tan\phi\tan\theta)\left(\frac{1-\tan^2\theta-2\tan\theta}{1-\tan^2\theta+2\tan\theta}\right)=0\ldots.(4)$. This is true because $\phi=22\frac{1}{2}°$.

Clearing (4) of fractions, resolving factors, and substituting for $\tan\phi=22\frac{1}{2}$ its equal $\sqrt{2}-1$, observing that $\cot\phi=\sqrt{2}+1$, we get $(b+a)\tan^3\theta+\{(b-a)(\sqrt{2}+1)+[2(b-a)]\}\tan^2\theta+\{2(b+a)(\sqrt{2}+1)-(b+a)\}\tan\theta=(b-a)(\sqrt{2}-1)\ldots.(5)$. Dividing (5) by $b+a$ and substituting the numerical values of a and b, we get

$\tan^3\theta-.490468\tan^2\theta+3.828427125\tan\theta=.268245951375$. Hence, by Horner's Method of Detached Coefficients, $\tan\theta=.0693633$, and the auxiliary angle $\theta=3°58'4\frac{7}{8}''$. By substituting in (1) and (2), we determine that $y=35.807338$ and $z=47.407325$. $\therefore$ $x=\sqrt{(y^2+z^2)}=59.410604$.

This problem was also solved by *A. H. Bell*, *J. F. W. Scheffer*, and *H. C. Wilkes*.

42. **Proposed by ALEXANDER MACFARLANE, A. M., D.Sc , LL. D., Cornell University, Ithaca, New York.**

There are p electors and q candidates for r seats. Each elector has r votes, and he may distribute them as he pleases among the candidates. Find in how many different ways the voting may result, that is, the number of possible states of the poll.

Solution by G. B. M. ZERR, Staunton, Virginia, and F. P .MATZ, New Windsor, Maryland.

The number of different ways of voting for r seats out of q candidates, when each elector casts r votes for r different persons, is

$$n=\frac{q(q-1)(q-2)(q-3)\ldots.(q-r+1)}{1.2.3.4\ldots.r}.$$

If $p>n$, then, since there can be but n different ways of voting, n will be the number of different ways the voting may result.

If $p<n$, then since p persons can prepare only p states of the poll, p will be the number of different ways the voting may result.

Also solved by *H. C. Whitaker.*

GEOMETRY.

Conducted by B. F. FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

40. Proposed by J. C. CORBIN, Pine Bluff, Arkansas.

If R, r, r_1, r_2, and r_3 be, respectively, the radii of the circumscribed, inscribed, and escribed circles of a $\triangle$, prove $r_1+r_2+r_3-r=4R$.

Solution by M. A. GRUBER, War Department, Washington, D. C.

From any $\triangle$ whose sides are a, b, and c, we obtain $R=\dfrac{abc}{4\triangle}$,

$$r=\frac{\triangle}{s},\ r_1=\frac{\triangle}{s-a},\ r_2=\frac{\triangle}{s-b},\ \text{and}\ r_3=\frac{\triangle}{s-c}.$$

$$\therefore\quad r_1+r_2+r_3-r=\frac{\triangle}{s-a}+\frac{\triangle}{s-b}+\frac{\triangle}{s-c}-\frac{\triangle}{s}=\frac{2s^3-as^2-bs^2-cs^2+abc}{\triangle}$$

$$=\frac{s^2[2s-(a+b+c)]+abc}{\triangle}=\frac{abc}{\triangle}.\quad \text{But}\ \frac{abc}{\triangle}=4R.\quad \therefore\quad r_1+r_2+r_3-r=4R.$$

We might appropriately add a few other combinations of these radii.

$$(1)\ \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r};\ (2)\ rr_1r_2r_3=\triangle^2;\ (3)\ Rrr_1r_2r_3=\frac{abc\triangle}{4}.$$

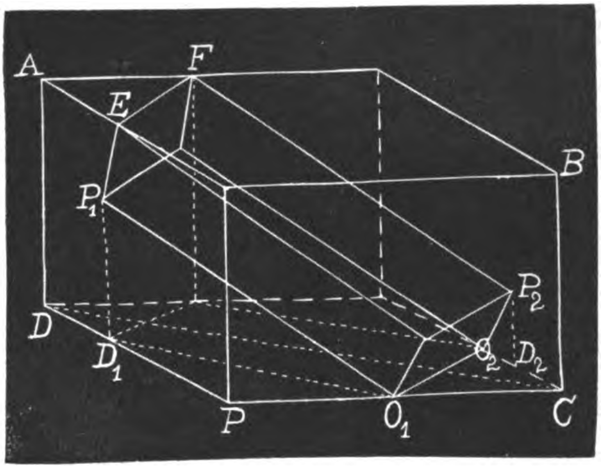
Solutions of this problem were received from *G. I. Hopkins*, *E. W. Morrell*, *P. S. Berg*, *G. B. M. Zerr*, *F. P. Matz*, *Cooper D. Schmitt*, *P. H. Philbrick*, *J. F. W. Scheffer*, *John B. Faught*, and the *Proposer*. H. C. Whitaker did not solve the problem but referred to Chauvenet's Geometry and Hallowell's Geometrical Analysis, p. 225.

41. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Find the length (x) of a rectangular parallelopiped $b=5$ft., and $h=3$ft., which can be *diagonally inscribed* in a similar parallelopiped $L=83$ft., $B=64$ft, and $H=50$ft.

Solution by B. F. BURLESON, Oneida, Castle, New York, and the PROPOSER.

Let $x=AF$, $y=AE$, $z=P_2D_2$, and $l=O_1P_1$ = the required length of the inscribed rectangular parallelopiped; then, obviously, $x^2+y^2=b^2$ ....(1), $(L-x)^2+(B-y)^2+(H-z)^2=l^2$ ....(2), $x(L-x)=y(B-y)$ .....(3), and $h_1[(L-x)^2+(B-y)^2]=lz$ ....(4).



From (3) and (1),

$$4y^4-4By^3+(B^2-4b^2+L^2)y^2+2Bb^2y=(L^2-b^2)b^2 \text{ ....(5)};$$

and this with coefficients numerically expressed, becomes

$$4y^4-256y^3+1088.5y^2+3200y=171600 \text{ .....(6)}.$$

Therefore, by *Horner's Method of Approximation*, we have from (6), $y=4$; whence $x=3$. Briefly putting the now known $(L-x)^2+(B-y)^2,=m^2=10000$, we have from (2) and (4), respectively, $m^2+(H-z)^2=l^2$ ....(7), and $lz=hm$ ....(8). Therefore, $l^4-(H^2+m^2)l^2+2Hhml=h^2m^2$ ....(9); that is, $l^4-12500l^2+30000l=90000$ ....(10).

Whence $l=110.6171303244 15$ feet.

COR.—Make $H=0$, and $h=0$; then the problem becomes: *Find the length of a rectangle of given width inscribed diagonally in a given rectangle.*

After performing obvious operations, we obtain

$$l^4-(B^2+2b^2+L^2)l^2+4BbLl=(B^2-b^2+L^2)b^2 \text{ ....(11)};$$

or with the coefficients numerically expressed, we have the equation,

$$l^4-11035l^2+106240l=274000 \text{ ....(12)}.$$

Therefore $l=100$ feet, which is the length of the diagonally-inscribed rectangle required.

A. H. Bell gets 107.5 feet as a result.

42. Proposed by G. I. HOPKINS, Instructor in Mathematics and Physics in High School, Manchester, New Hampshire.

If the bisectors of two angles of a triangle are equal the triangle is isosceles.

Solution by the PROPOSER.

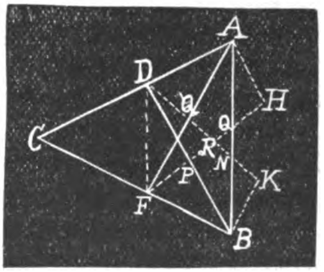
Let AF and BD bisect the angles of the triangle ABC, and let $AF=BD$.

Draw DF. Make $\angle PDO=\angle PDF$, and $\angle QFN=\angle QFD$.

Draw AH perpendicular to AF and BK perpendicular to BD.

Draw FH through O and DK through N.

$\triangle DFB = \angle DOB$, having two angles and included side of one etc. $\therefore$ $BF = BO$. $\therefore$ BP is perpendicular to OF, for a line which bisects the vertical angle of an isosceles triangle is perpendicular to the base. Similarly $AD = AN$, and AQ is perpendicular to ND. $\triangle$'s DPR and FQR are right-angled at P and Q. $\therefore$ $\angle RDP = \angle QFR$. $\therefore$ $\triangle$'s AHF and KBD are equal, since they are right-angled at B and A, and have a leg and adjacent acute angle of one equal respectively to a leg and adjacent acute angle of the other.



$\therefore$ $AH = KB$. BK is parallel to HF, and AH is parallel to KD, being perpendicular to the same line. $\therefore$ $\angle KBN = \angle HOA$, and $\angle KNB = \angle HAO$, being exterior interior angles. $\therefore$ $\angle H = \angle K$.

$\therefore$ $\triangle$'s KNB and HAO are equal, having two angles and included side etc. $\therefore$ $AO = NB$. $\therefore$ $AN = OB$. $\therefore$ $AD = BF$. $\therefore$ $\triangle$'s ADF and BDF are equal, having three sides respectively equal.

$\therefore$ $\angle DAF = \angle DBF$, and $\therefore$ $\angle A = \angle B$. $\therefore$ $AC = BC$, being opposite equal angles.

Q. E. D.

As this problem is one that has frequently been discussed and is of interest to mathematicians we shall publish, in the June MONTHLY, two or three more of the many excellent solutions we have received. A query from Dr. George Lilley says, "It is said that Mr. I. Todhunter proposed the above problem, and that a direct or *a priori* proof has not been discovered for it. What is the *a priori* proof?—ED.

PROBLEMS.

46. Proposed by GEORGE E. BROCKWAY, Boston, Massachusetts.

If an equilateral triangle is inscribed in a circle, the sum of the squares of the lines joining any point in the circumference to the three vertices of the triangle is constant.

47. Proposed by J. C. GREGG, Superintendent of Schools, Brazil, Indiana.

Given two points A and B and a circle whose center is O: show that the rectangle contained by OB and the perpendicular from B on the polar of A is equal to the rectangle contained by OB and the perpendicular from A on the polar of B.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

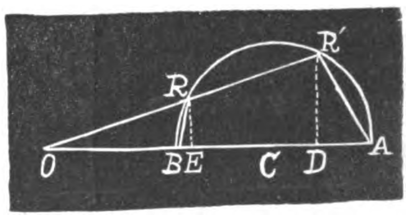
SOLUTIONS OF PROBLEMS.

31. Proposed by E. B. ESCOTT, Ann Arbor, Michigan.

Through a point O on the produced diameter AB of a semicircle draw a secant ORR', so that the quadrilateral $ABRR'$ inscribed in the semicircle shall be a maximum. Prove that in this case, the projection of RR' on AB is equal in length to the radius of the circle. [*Williamson's Diff. Calculus*, 7th edition. p. 189, Ex. 25.]

I. Solution by JOHN B. FAUGHT, A. B., Instructor in Mathematics, Indiana University, Bloomington, Indiana.

Let $OC=p$; then the equation of the circle is $\rho^2-2\rho p\cos\theta+p^2-r^2=0$. The roots of which are $\rho_1=p\cos\theta+\sqrt{(r^2-p^2\sin^2\theta)}=OR'$ and $\rho_2=p\cos\theta-\sqrt{(r^2-p^2\sin^2\theta)}=OR$.



$\triangle OAR'=OA.OR'\sin\theta=(p+r)[p\cos\theta+\sqrt{(r^2-p^2\sin^2\theta)}]\sin\theta$.

$\triangle OBR=AB.OR\sin\theta=(p-r)[p\cos\theta-\sqrt{(r^2-p^2\sin^2\theta)}]\sin\theta$.

$\therefore$ Qd. $ABRR'=2p[r\sin\theta\cos\theta+\sin\theta\sqrt{(r^2-p^2\sin^2\theta)}]$.

$$\therefore \frac{d(\text{Qd})}{d\theta}=\frac{r(1-2\sin^2\theta)\sqrt{(r^2-p^2\sin^2\theta)}+\cos\theta(r^2-2p^2\sin^2\theta)}{\sqrt{(r^2-p^2\sin^2\theta)}}=0, \text{ for a}$$

maximum or minimum.

Reducing we have $4p^2\sin^4\theta-4(p^2+r^2)\sin^2\theta+3r^2=0$.

$$\therefore \sin\theta=\frac{\sqrt{(r^2+p^2)\pm\sqrt{(r^4-r^2p^2+p^4)}}}{p},$$ which must be a maximum

from the nature of the problem.

$$ED=OR'\cos\theta-OR\cos\theta=(\rho_1-\rho_2)\cos\theta=2\sqrt{(r^2-p^2\sin^2\theta)}\cos\theta$$

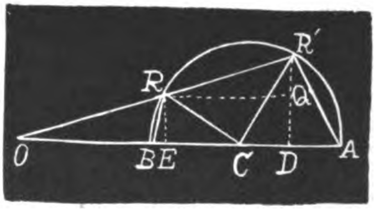
$$=\sqrt{4(r^2-p^2\sin^2\theta)(1-\sin^2\theta)}=\sqrt{4r^2-4(p^2+r^2)\sin^2\theta+4p^2\sin^4\theta}$$

$$=\sqrt{(4r^2-3r^2)}=r.$$

II. Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland, and J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

Let $CB=r$, $CO=a$, $a/r=e$, $\angle COR=\theta$, $CR'O=\phi$; then $\angle ACR'=(\phi+\theta)$, $\angle OCR=(\phi-\theta)$, $OR'=r\sin(\phi+\theta)/\sin\theta$, $OR=r\sin(\phi-\theta)/\sin\theta$, and

$\sin\phi = c\sin\theta \ldots (1)$. The area of the quadrilateral is $\mathbf{A} = \frac{1}{2}[AO + OR' - BO \times OR]\sin\theta, = ar[\cos\theta + \sqrt{(1-c^2\sin^2\theta)}]\sin\theta \ldots (2)$, which is to be a maximum.



Differentiating, etc., we have

$$\frac{d\mathbf{A}}{d\theta} = \frac{1-2\sin^2\theta}{1-2c^2\sin^2\theta} + \frac{\sqrt{(1-\sin^2\theta)}}{\sqrt{(1-c^2\sin^2\theta)}} = 0 \ldots (3).$$

$\therefore\ 4c^2\cos^4\theta - 4(c^2-1)\cos^2\theta - 1 = 0 \ldots (\alpha)$,

or $4c^2\sin^4\theta - 4(c^2+1)\sin^2\theta + 3 = 0 \ldots (\beta)$.

From (α), $\cos^2\theta = \frac{1}{2}c^{-2}\{\sqrt{[(c^2-1)^2+c^2]} + (c^2-1)\} \ldots (4)$.

From (1), $\cos^2\phi = \frac{1}{2}\{\sqrt{[(c^2-1)^2+c^2]} - (c^2-1)\} \ldots (5)$.

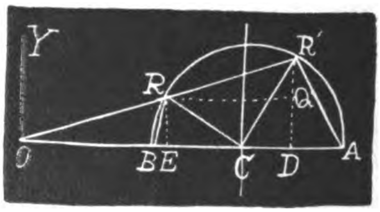
The projection of RR' on AB is $\mathbf{P} = (OR' - OR)\cos\theta, = 2r\cos\phi\cos\theta \ldots (6)$.

Substituting the square roots of (4) and (5) in (6), etc., we have $\mathbf{P} = r$.

III. **Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.**

Let O be the origin, Ox, Oy the axes,, C the centre of the circle, $OC = a$, $BC = AC = r$, $\tan ROB = m$.

Then equation to circle is $(x-a)^2 + y^2 = r^2$, and equation to line ORR' is $y = mx$.



By substitution we get

$$(x-a)^2 + m^2x^2 = r^2.$$

$$\therefore\ x = \frac{a \pm \sqrt{(r^2 + r^2m^2 - a^2m^2)}}{1+m^2},$$

$$y = \frac{ma \pm m\sqrt{(r^2 + r^2m^2 - a^2m^2)}}{1+m^2}.$$

Now area $OR'A$ − area ORB = max. $\therefore\ \frac{1}{2}OA \times R'D - \frac{1}{2}OB \times RE$ = max.

$$\therefore\ \tfrac{1}{2}(a+r)\left\{\frac{ma + m\sqrt{(r^2+r^2m^2-a^2m^2)}}{1+m^2}\right\}$$

$$-\tfrac{1}{2}(a-r)\left\{\frac{ma - m\sqrt{(r^2+r^2m^2-a^2m^2)}}{1+m^2}\right\} = \text{max}.$$

$$\therefore\ \frac{mar + ma\sqrt{(r^2+r^2m^2-a^2m^2)}}{1+m^2} = \text{max}.$$

Reducing the first differential coefficient we get

$$r^2 + r^2m^2 - 2a^2m^2 = r(m^2-1)\sqrt{(r^2+r^2m^2-a^2m^2)}.$$

Squaring and reducing we easily get $m^4r^2 + 2(2a^2 - r^2)m^2 = 3r^2$.

$\therefore\ m^2 = \frac{1}{r^2}[2\sqrt{(a^4 - a^2r^2 + r^4)} - 2a^2 + r^2]$, as the only admissible value.

The projection of RR' on $AB = ED = OD - OE = \dfrac{2\sqrt{(r^2+r^2m^2-a^2m^2)}}{1+m^2} = r$ for the above value of m^2.

When $a = r$, $m^2 = 1$. $\therefore\ \angle ROB = 45°$.

Also solved by *Alfred Hume*, *C. E. White*, and *H. W. Draughon*.

MECHANICS.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

17. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Find the law of density of strings collected into a heap at the edge of a table with the end of the string just over the edge, so that equal masses may always pass over in equal units of time.

II. Solution by the PROPOSER.

Let x=the length of string which depends from the edge of the table at the end of the time t from the beginning of motion, k=the density of the string at a unit's distance from the end, and *assume* that the density varies as the nth power of the distance from the end. The mass of the depending length is then $=\int_0^x kx^n dx=\frac{k}{n+1}x^{n+1}$, and if a= the mass passing over the edge in a unit of time, and g=the acceleration of gravity, we have for the equation of motion, $\frac{d}{dt}\left(\frac{k}{n+1}x^{n+1}\frac{dx}{dt}\right)=gat\ldots.(1)$,

$$\text{or, } \frac{d}{dt}\left(\frac{k}{n+1}x^{n+1}\frac{dx}{dt}\right)=\frac{k}{n+1}gx^{n+1}\ldots.(2),$$

or, $\frac{d}{dt}\left(x^{n+1}\frac{dx}{dt}\right)=gx^{n+1}\ldots.(3)$.

Multiplying both members of (3) by $2\left(x^{n+1}\frac{dx}{dt}\right)$ and integrating,

$$\left(x^{n+1}\frac{dx}{dt}\right)^2=\frac{2g}{2n+3}x^{2n+3}\ldots.(4), \text{ or, } \frac{dx^2}{dt^2}=\frac{2g}{2n+3}x\ldots.(5).$$

Equation (5) gives $\frac{d^2x}{dt^2}=\frac{g}{2n+3}\ldots.(6)$,

$$\text{whence } \frac{dx}{dt}=\frac{g}{2n+3}t\ldots.(7).$$

From (7), $gat=a(2n+3)\frac{dx}{dt}=\frac{d}{dt}\left(\frac{k}{n+1}x^{n+1}\frac{dx}{dt}\right)\ldots.(8)$.

Integrating (8) and dividing by x, $\frac{k}{n+1}n^n\frac{dx}{dt}=a(2n+3)\ldots.(9)$.

Equation (9) gives $\frac{1}{t}\cdot\frac{k}{n+1}x^{n+1}=a(n+1)(2n+3)=a\ldots.(10)$,

since the *mass* of string passes over at the uniform rate a.

Equation (10) gives $2n^2+5n=-2$, or $n=-\frac{1}{2}$, or $n=-2$.

19. Proposed by H. C. WHITAKER, B. Sc., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

"There was an old woman tossed up in a basket,
Ninety times as high as the moon."
What was her initial velocity, the resistance of the air being neglected?

Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Mississippi, and J. C. NAGLE, M. A., C.E., Professor of Civil Engineering, A and M. College, College Station, Texas.

If g is the acceleration of gravity at the earth, R the surface of the earth's mean radius, and x the distance of the body from the center of the earth at the time t, the equation of motion is $\dfrac{d^2x}{dt^2}=-\dfrac{a^2}{x^2}g$.

Integrating, $\left(\dfrac{dx}{dt}\right)^2=\dfrac{2a^2}{x}g+c$.

Taking the moon's distance as $60.3R$, $\dfrac{dx}{dt}=0$ when $x=5427R$.

$$\therefore\quad c=-\frac{2R^2}{5427R}g,\quad\text{and}\quad\left(\frac{dx}{dt}\right)^2=2R^2g\left(\frac{1}{x}-\frac{1}{5427R}\right).$$

When $x=R$, the velocity (the initial velocity required)

$$=\sqrt{2Rg\left(\frac{1}{R}-\frac{1}{5427R}\right)}=\sqrt{\frac{5399}{2713.5}}\,Rg,\ \text{which is } 6.9+ \text{ miles per second.}$$

Also solved by *P. S. Berg*, *E. W. Morrell*, *H. W. Draughon*, *G. B. M. Zerr*, *F. P. Matz*, and the *Proposer*.

NOTE.—Professor Hoover sent a fine solution of problem 18, but it came to late for insertion in April MONTHLY.—EDITOR.

PROBLEMS.

27. Proposed by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

One thousand balls, each having a mass of 10 grams, and each moving with a velocity of 10 kilometers per second, are confined in a certain space with elastic walls. Into the same space are now introduced one thousand balls each of 100 grams mass, and moving with a velocity each of 10 kilometers per second; collisions take place, and finally, after a number of encounters, the average kinetic energy of each of the two thousand balls is the same. Show that this is 2.75(10)11 in the centimeter-gram-system.

28. Proposed by O. W. ANTHONY, Mexico, Missouri.

A movable finite wire carrying a current os electricity is perpendicular to and on one side of an infinite wire also carrying a current. Invertigate the motion of the movable wire.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

21. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, in New Windsor College, New Windsor, Maryland.

Find (1) nine positive *integral numbers* in arithmetical progression the sum of whose squares is a *square number*; and (2) find nine integral *square numbers* whose sum is a *square number*.

Solution by ARTEMAS MARTIN, LL.D., U. S. Coast and Geodetic Survey Office, Washington, D.C.

1. Let $x-4y$, $x-3y$, $x-2y$, $x-y$, x, $x+y$, $x+2y$, $x+3y$, $x+4y$ denote the required numbers in arithmetical progression, and the sum of their squares is $9x^2+60y^2=\square\ \ldots.(1)$.

Put $y=3z$ and (1) becomes, after dividing by 9,

$$x^2+60z^2=\square=\left(x+\frac{p}{q}z\right)^2\ \ldots.(2);$$

$$\text{whence } \frac{x}{z}=\frac{60q^2-p^2}{2pq}.$$

Let $p=1$, $q=\frac{1}{2}$; then $\frac{x}{z}=\frac{14}{1}$. Take $x=14$, $z=1$; then $y=3$, and the numbers are 2, 5, 8, 11, 14, 17, 20, 23, 26; and we have

$$2^2+5^2+8^2+11^2+14^2+17^2+20^2+23^2+26^2=48^2.$$

Let $p=1$, $q=1$; then $\frac{x}{z}=\frac{59}{2}$. Take $x=59$, $z=2$; then $y=6$, and the numbers are 35, 41, 47, 53, 59, 64, 70, 76, and 80; and we have

$$35^2+41^2+47^2+53^2+59^2+64^2+70^2+76^2+82^2=183^2.$$

By giving suitable values to p and q an infinite number of sets may be found.

2. Take the well-known identity $(x+y)^2=(x-y)^2+4xy\ \ldots.(3)$.

If we can transform $4xy$ into a square we shall have two square numbers whose sum is a square. Since x may be any quantity whatever, we may put $x=a+b+c+d+e+f+g+h$, and then we have

$$(a+b+c+d+e+f+g+h+y)^2=(a+b+c+d+e+f+g+h-y)^2$$
$$+4y(a+b+c+d+e+f+g+h)\ \ldots.(4).$$

The last term will be a square if we take $a=i^2$, $b=j^2$, $c=k^2$, $d=l^2$, $e=m^2$, $f=n^2$, $g=p^2$, $h=q^2$, $y=r^2$, and we have

$$(i^2+j^2+k^2+l^2+m^2+n^2+p^2+q^2+r^2)^2$$
$$=(i^2+j^2+k^2+l^2+m^2+n^2+p^2+q^2-r^2)^2.$$

$+(2ri)^2+(2rj)^2+(2rk)^2+(2rl)^2+(2rm)^2+(2rn)^2+(2rp)^2+(2rq)^2 \ldots (5)$. Take $i=1)$, $j=2$, $k=3$, $l=4$, $m=5$, $n=6$, $p=7$, $q=8$, $r=9$; then, after dividing the numbers by 3, we have $6^2+12^2+18^2+24^2+30^2+36^2+41^2+42^2+48^2=95^2$.

An infinite number of sets of nine square numbers whose sum is a square may be found from (5).

27. Proposed by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Decompose into the sum of two squares the product $5\times13\times61$.

I. Solution by DAVID E. SMITH, Ph. D., Professor of Mathematics, State Normal School, Ypsilanti, Michigan.

Taking the usual formula

$$(m^2+n^2)(x^2+y^2)=(mx\pm ny)^2+(my\mp nx)^2,$$ and noticing that $5.13.61=65.61$, the problem reduces to decomposing 65 and 61 into two squares, which give

$$65=64+1=49+16,$$
$$61=36+25.$$
$$\therefore\quad 65.61=53^2+34^2=43^2+46^2=62^2+11^2=22^2+59^2.$$

The products 1.3965, 5.793, and 13.305 give no different results.

II. Solution by the PROPOSER.

$$(m^2+n^2)(p^2+q^2)=(mp\pm nq)^2+(mq\mp np)^2, =A^2+B^2.$$
$$(m^2+n^2)(p^2+q^2)(r^2+s^2)=(A^2+B^2)(r^2+s^2), =(Ar\pm Bs)^2+(As\mp Br)^2.$$

Let $m=2$, $n=1$, $p=3$, $q=2$, $r=5$, $s=6$.

Then $A^2+B^2=(6\pm2)^2+(4\mp3)^2, =8^2+1^2, =4^2+7^2$.

$$(2^2+1^2)(3^2+2^2)(5^2+6^2)=5\times13\times61$$
$$=(40\pm6)^2+(48\pm5)^2, =(20\pm42)^2+(24\mp35)^2.$$
$$=46^2+43^2, =34^2+53^2, =62^2+11^2, =22^2+59^2.$$

III. Solution by COOPER D. SCHMITT, Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

I present a general solution of this problem, true for any number found according to the same law.

$$5=1^2+2^2,\ 13=(1+2)^2+(1\times2)^2, =3^2+2^2,\ 61=(3+2)^2+(3\times2)^2=5^2+6^2.$$

Then in general we have $(x^2+y^2)[(x+y)^2+(xy)^2][(x+y+xy)^2+(xy)^2(x+y)^2]$ and this is to be shown to consist of two squares.

Using determinants we have as follows:

$$\begin{vmatrix} x & -y \\ y & x \end{vmatrix} \times \begin{vmatrix} x+y & -xy \\ xy & x+y \end{vmatrix} \times \begin{vmatrix} x+y+xy & -x^2y-xy^2 \\ x^2y+xy^2 & x+y+xy \end{vmatrix}$$

$$= \begin{vmatrix} x^2+xy+xy^2, & x^2y-xy-y^2 \\ xy+y^2-x^2y, & xy^2+x^2+xy \end{vmatrix} \times \begin{vmatrix} x+y+xy & -x^2y-xy^2 \\ x^2y+xy^2 & x+y+xy \end{vmatrix}$$

$$\left| = \begin{matrix} (x^2+xy+xy^2)(x+y+xy)+(x^2y-xy-y^2)(-x^2y-xy^2), \\ (xy+y^2-x^2y)(x+y+xy)+(xy^2+x^2+xy)(-x^2y-xy^2), \end{matrix}\right.$$

$$\left.\begin{matrix} (x^2+xy+xy^2)(x^2y+xy^2)+(x^2y-xy-y^2)(x+y+xy) \\ (xy+y^2-x^2y)(x^2y+xy^2)+(xy^2+x^2+xy)(x+y+xy) \end{matrix}\right|$$

$$=(x^3+2x^2y+2x^2y^2+xy^2+xy^3+x^3y+3x^2y^3+xy^4+x^3y^2-x^4y^2-x^3y^3)^2$$

$$+(x^4y+3x^3y^2+x^3y^3+x^2y^3+x^2y^4+x^3y-x^2y-2xy^2-y^3-xy^3)^2.$$

If $x=1$, $y=2$, as in given problem, I get $5\times13\times61=59^2+22^2$.

If $x=2$, $y=3$, I find $13\times61\times2021=398^2+807^2$ and so on indefinetly.

AVERAGE AND PROBABILITY.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

14. Proposed by CHARLES E. MYERS, Canton, Ohio.

$\frac{1}{3}$ of all the melons in a patch are not ripe, and $\frac{1}{4}$ of all the melons in the same patch are rotten, the remainder being good. If a man enters the patch on a dark night and takes a melon at random, what is the probability that he will get a good one?

I. Comment by JOHN DOLMAN, Jr., Counsellor-at-Law, Philadelphia, Pennsylvania.

The published solution of Probability problem No. 14 is erroneous. The simplest correct solution is as follows:

If the melon selected is *not unripe* and *not rotten*, it will be good. The chance that it is unripe is $\frac{1}{3}$, therefore the chance that it is not unripe is $\frac{2}{3}$; by similar reasoning, the chance that it is not rotten is $\frac{3}{4}$. Therefore the chance that it is not unripe and not rotten, is $\frac{2}{3}$ of $\frac{3}{4}=\frac{1}{2}$, which is the chance required.

It is not very difficult to point out the error in the published solution. While it is true there cannot be more than $8n$ nor less than $5n$ good melons, it does not follow that $\frac{1}{2}(8n+5n)$ is the average or most probable number, unless it be predicated that all values between $5n$ and $8n$ are equally likely, which is not the case.

Dropping the n's, suppose there are 8 ripe and 4 unripe melons, the three rotten ones may be selected from these 12 in 220 different ways each of equal probability. Now a very simple application of the principles of choice will show that of these ways

4 would leave 8 good melons
48 would leave 7 good melons
112 would leave 6 good melons
56 would leave 5 good melons

giving therefore the numbers 5, 6, 7, and 8 the relative values thus found, and averaging in the ordinary way, we will find that 6 is the average number of good melons, and therefore $\frac{6}{18}$ or $\frac{1}{3}$ is the probability of selecting a good one.

15. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, in New Windsor College, New Windsor, Maryland.

Todhunter proposes: "From a point in the circumference of a circular field a projectile is thrown at random with a given velocity, which is such that the diameter of the field is equal to the greatest range of the projectile: prove the chance of its falling within the field, is $C=2^{-1}-2\pi^{-1}(\sqrt{2}-1),=.236+$." Is this result perfectly correct as to fact?

Comment on the Solution of Problem 15, by JOHN DOLMAN Jr., Philadelphia, Pennsylvania.

I should not presume to criticise the work of so able and celebrated a mathematician as Professor Matz, did I not consider his solutions of Probability No. 15 vicious in their effects upon the minds of students, striking as they do at the root, not only of the doctrine of mean value and probability, but of the integral calculus itself. "Since the projectiles are *thrown* at random they should *fall* at random," is on a par with, "as an arc varies uniformly the sine varies uniformly," or "because acceleration is constant velocity is constant." The third solution is not clear, but how far any given range the favorable chances can be represented by an area, when the projectiles must fall on the arc of a circle, is difficult to understand. Also in the fifth solution it is stated," for any range PD', the projectiles falling on the circular arc DMD' are within the field" and then the angle PAD' is adopted as uniformly varying, without giving any reason for it. However interesting this may be as mathematical legerdemain, its effects are vicious when it is published without proper explanation for in my humble opinion it is more important that your readers learn to reason correctly than that they be taught to integrate ingeniously.

NOTE.—The solution of problem 14 was published without comment for the reason that we considered the solution to be correct, and we confess that we do not yet see the force of Mr. Dolman's argument, though we have not had time to give it much thought.

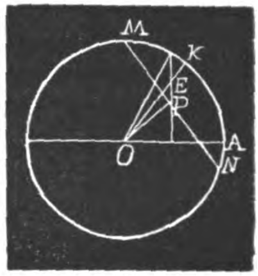
As to the solutions of problem 15, we hold that the first solution is the only correct solution as that one and that one alone involves the strict literal statement of the problem. It is evident that the number of ways the projectile can be thrown is equal to the surface of a hemisphere whose radius is R. If now we find the surface of that part of this hemisphere any point at which if a projectile be thrown the projectile will fall upon the circular field, diameter R, and then divide this surface by the surface of the hemisphere, the result will be the probability required. This method of solution would have to be accepted by the most critical mind. Professor Matz's first solution involves this principle.—EDITOR.

19. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Find the average area of the circle which is the locus of the middle points of all chords passing through a point taken at random in the surface of a given circle.

I. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let P be the random point. Through P draw HG perpendicular to OA. Let $OA=a$, $GP=x$, $\angle HOA=\theta$, area circle required$=\frac{\pi}{4}OP^2=\frac{\pi}{4}(x^2+a^2\cos^2\theta)$. An element of the circle at P, is $a\sin\theta d\theta dx$. The limits of θ are 0 and 2π; of x, 0 and $a\sin\theta=x'$. $\triangle$=average area.



$$\therefore\ \triangle=\frac{\int_0^{2\pi}\int_0^{x'}\frac{\pi}{4}(x^2+a^2\cos^2\theta)a\sin\theta d\theta dx}{\int_0^{2\pi}\int_0^{x'}a\sin\theta d\theta dx}$$

$$=\frac{1}{4a}\int_0^{2\pi}\int_0^{x'}(x^2+a^2\cos^2\theta)\sin\theta d\theta dx=\frac{a^2}{12}\int_0^{2\pi}(\sin^2\theta+3\cos^2\theta)\sin^2\theta d\theta=\frac{\pi a^2}{8}.$$

II. Solution by the PROPOSER.

Let $OP=x$; then the average area of the circle whose diameter is OP, becomes $A=\int_0^{2\pi}\int_0^r(\frac{1}{4}\pi x^2)xd\theta dx\div\int_0^{2\pi}\int_0^r xd\theta dx=\frac{1}{8}\pi r^2$.

Professor Zerr furnished five different solutions of this problem; Professor Matz seven and Mr. Dolman three.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

15. **Proposed by SAMUER HART WRIGHT, M.D., M.A.,Ph.D., Penn Yan, Yates Co., New York.**

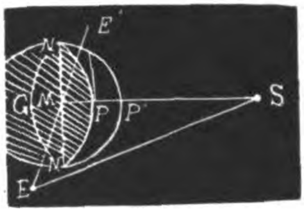
Required the illuminated area of the Moon's disc when $\frac{2}{3}$ through its first quarter of 60° of longitude east of the Sun, the Earth and Moon being at their mean distances.

II. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia, and the PROPOSER.

A rigorous solution of this problem would present many difficulties. In the first place, the surface of the Moon is more than half illuminated; in the second, no observer sees half of the Moon's surface at one time, in the third, the mean distance of the Sun is known within a quarter million miles.

In the following solution we will assume half the Moon's surface illuminated, and half of its surface as presented to an observer at one time. Also we will take the Sun's parallax 8.″81 and hence, his mean distance as 149320000 kilometers, and the Moon's mean distance as 384000 kilometers.

Let S, M, E, be the centres of the Sun, Moon, and Earth respectively. Let $EM=1$, then $SE=388.854$, also $\angle MES=60^\circ$.



From Trigonometry we get

$EM+ES:ES-EM=\tan\frac{1}{2}(EMS+ESM)$
$:\tan\frac{1}{2}(EMS-ESM)$
$389.854:387.854=\tan 60^\circ:\tan\frac{1}{2}(EMS-ESM)$.

$\therefore\ \frac{1}{2}(EMS-ESM)=59^\circ 52' 20''$,

$\therefore\ \angle EMS=119^\circ 52' 20''$.

Now since EM is perpendicular to the plane $N'BNP'$, and MS is perpendicular to the plane $N'GNP$, $\angle P'ME=180^\circ-119^\circ 52' 20''=60^\circ 7' 40''=\theta$.

Let r=radius of the Moon, then MP the semi-conjugate diameter of the ellipse $N'GNP=r\cos\theta$.

Area of bright crescent $N'PNP'N'$=area of semi-ellipse $NMNPN'$ subtracted from semi-circle $N'NPN'$; area semi-circle$=\frac{1}{2}\pi r^2$,

area semi ellipse$=\frac{1}{2}\pi r^2\cos\theta$; and area crescent$=\frac{1}{2}\pi r^2(1-\cos\theta)=.250965\pi r^2$ $=\frac{1}{4}$ of the disc nearly. In the above solution we have regarded the observer as being at the centre of the earth.

16. Yale Senior Prize Problem. Contributed by H. A. NEWTON, LL.D., Professor of Mathematics, Yale College, New Haven, Connecticut.

The axis of two right cylinders whose bases are circles of 4 and 6 inches radius respectively, intersect at right angles: compute to four decimal places the lengths of the curves of intersection of the two surfaces.

Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let $x^2+y^2=36$, be the equation to the cylinder with base radius 6 inches.

$x^2+z^2=16$, the equation to the cylinder with base radius 4 inches.

$$\text{Then}\ \frac{dy}{dx}=-\frac{x}{\sqrt{(36-x^2)}},\ \frac{dz}{dx}=-\frac{x}{\sqrt{(16-x^2)}}.$$

$$\therefore\ S=\int\left\{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2\right\}^{\frac{1}{2}}dx=\int\left\{\frac{576-x^4}{(36-x^2)(16-x^2)}\right\}^{\frac{1}{2}}dx.$$

$$\text{The whole length of the curve}=4s=4\int_0^4\left\{\frac{576-x^4}{(36-x^2)(16-x^2)}\right\}^{\frac{1}{2}}dx.$$

Let $x=4\sin\theta$.

$$\text{Then}\ 4s=16\int_0^{\frac{\pi}{2}}\left\{\frac{9-4\sin^4\theta}{9-4\sin^2\theta}\right\}^{\frac{1}{2}}d\theta=16\int_0^{\frac{\pi}{2}}\left\{\frac{1-e^2\sin^4\theta}{1-e^2\sin^2\theta}\right\}^{\frac{1}{2}}d\theta,\ \text{where}\ e=\tfrac{2}{3}.$$

Expanding we get

$$4s=16\int_0^{\frac{\pi}{2}}(1+\tfrac{2}{9}\sin^2\theta-\tfrac{4}{27}\sin^4\theta-\tfrac{16}{729}\sin^6\theta-\tfrac{200}{6561}\sin^8\theta-\tfrac{448}{6561}\sin^{10}\theta$$
$$-\tfrac{1408}{177147}\sin^{12}\theta-\tfrac{3302}{1594323}\sin^{14}\theta-\ldots.)d\theta.$$

$4s=8\pi(1+\frac{1}{9}-\frac{1}{18}-\frac{5}{729}-\frac{875}{104976}-\frac{7}{3888}-\frac{847}{472392}-\frac{53053}{119042784}-\ldots.)$.

$4s=26.0104$ exact to two decimal places.

Also solved by *F. P. Matz*, and *J. F. W. Scheffer*.

PROBLEMS.

30. Proposed by R. J. ADCOCK, Larchland, Illinois.

When the sum, of the distances of a point of a plane surface, from all its other points, is a minimum, that point is the centre of gravity of the plane surface.

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

"*The Mysterious Formula.*"

Referring to my article on "Logarithms of Negative Numbers" published in the April Number, Vol. I., Mr. C. D. Schmitt, on page 214, deduces the following singular result: $\pi=\dfrac{\log(-1)}{\sqrt{-1}}$.

Another very remarkable result can be deduced from this as follows:

Dividing by 2, we have $\dfrac{1}{2\sqrt{-1}}\log(-1)=\frac{1}{2}\pi$. This may be written $\log\sqrt{-1}^{-\sqrt{-1}}=\frac{1}{2}\pi$. $\therefore$ $\sqrt{-1}^{-\sqrt{-1}}=e^{\frac{1}{2}\pi}=4.810477381$. This is what Professor Benjamin Peirce in his linear Associative Algebra, p. 5 (edition published by D. Van Nostrand), calls "the mysterious formula."

Writing i for $\sqrt{-1}$, the formula is $\frac{1}{i}=\sqrt{e^{\pi}}$.

M. C. STEVENS.

Editor Finkel has notified me that he cannot spare the space for further discussion of the possibility of a root of the equation

$4+\sqrt{(x-4)}-\sqrt{(x+4)}=0$. I will therefore again refer Mr. Draughon to Mr. Horner's article in the Philosophical Magazine, and also to Wentworth's Algebraic Analysis, page 278 to page 286.

H. C. WHITAKER.

The International Mathematical Congress.

Professor A. Vasiliev, President of the Physico-mathematical Society of Kasan, Russia, has sent me a document prepared by him for the minister of public instruction, with a request that I translate such part of it from the Russian as bears on the founding of an International Mathematical Congress, and make it known in America. This is in substance as follows:

After recapitulating the action of the French Association for the advancement of Science at Caen (August 14, 1894) [already translated by me and published on pp. 21-22 of the Bulletin of the American Mathematical Society, October 1894], he gives the resolution offered by me that very same day, August 14, 1894, for their signatures to all of the members of the American Mathematical Society present at the Brooklyn meeting, and signed unanimously, which was as follows: "The undersigned, members of the American Mathematical Society present at its summer meeting, 1894, take this method of expressing their cordial approval of a series of international Congresses of mathematicians to take place from time to time, as suggested by A. Vasiliev and C. A. Laisant." The names of the signers may be found on page 290 of of Vol. I of the MONTHLY. I explained the plan as contemplating a *reunion prepartoire* at Kasan in 1896, and a *congres constituant* in Belgium or Switzerland in 1897, which perhaps might fix the first international congress at Paris in 1900.

Professor Vasiliev then goes on to state the decisive step taken by the *deutsche Mathematiker-Vereinigung* in a reunion at Vienna, September 1894. It was there unanimously resolved to take part in the organizing congress. The action was as follows: "Concerning future international congresses, the Mathematiker-Vereinigung decides in principle to participate, and charges its bureau to take in regard to this subject the measures that appear necessary. In particular, it leaves to each of its members entire freedom, considering alone as essential that the socieiy, on this important occasion, may be assured of having the place due it." Professor Vasiliev expects that the inauguration of the Lobachevsky monument at Kasan will take place in August or September 1896, and counts on having there a large number of eminent mathematicians, and will profit by the occasion to propose definitely the organisation of the international congress, and then official calls will be issued to meet for the purpose of final organization in 1897 at a city of Belgium or Switzerland.

GEORGE BRUCE HALSTED.

Austin, Texas.

1. What explanation do the mathematicians of the *present* day give of the old paradox following, where precisely opposite results are reached by apparently rigorous demonstrations?

If A and B are traveling in the same line, A at the rate of one mile an hour and B, starting one mile behind him and traveling at the rate of two miles per hour, it is evident that in one hour A will have traveled one mile and B, two miles and will have overtaken A, as we know by experience the truth is. But while B is traveling one mile, A will have traveled one half as far, and so on. That is, while B is traveling the space between him and A at the commencement of an hour, A will have traveled half as far, and hence B can never overtake him.

To the human mind, in the absence of all other knowledge, this demonstration is rigorously correct and conclusive. But we know that the conclusion is absolutely the opposite of the truth. The only explanation formerly given was, that while the human mind cannot comprehend it, the space between A and B finally becomes infinitely small, or, as compared with finite quantities, *nothing*.

J. H. DRUMMOND.

II. It is often stated that it is impossible to trisect an angle with the rule and compass only. Has this impossibility been demonstrated and if so where can the demonstration be found?

W. E. HEAL.

EDITORIALS.

We shall consider it a great favor if all subscribers who have not yet paid their subscription for 1895, will kindly remit at once. We need the money to pay the Publishers.

At a recent meeting of the Philosophical Faculty of Yale University, Editor Finkel was assigned a graduate Scholarship for the academical year of 1895-6.

Rudyard Kipling will shortly return to India where he will prepare, for *The Cosmopolitan*, twelve articles to appear in the American and English editions of that magazine. India is one of the most interesting of countries, and Mr. Kipling is able to write of it as no one else. His work will be looked forward to with world-wide expectation.

Leonard E. Dickson, of the Chicago University, was reappointed to a Fellowship and will remain at the University another year.

There are a number of our subscribers in arrears for 1894. We shall consider it a kindness if those who are owing for 1894, will remit the amount of the subscription at once. A mathematical journal of the size and scope of the MONTHLY can not be published without funds, and were it not for a number of our mathematical friends aiding us financially the MONTHLY would be obliged to discontinue.

Subscribers who wished their subscription to cease with Vol. I. should have notified us on the receipt of Dec. No. Some after receiving three or four extra numbers, ask us to discontinue, without paying for the extra copies.

Our mathematics does not teach any such principles and should we find among our mathematical collections a book teaching any such doctrine we would consign the same to the flames. One of the very best things that can be claimed for the study of mathematics is that it develops strong tendencies to honesty and justice.

BOOKS AND PERIODICALS.

The Cosmopolitan: An Illustrated Monthly Magazine. Edited by John Brisben Walker and Arthur Sherburne Hardy. Price, $1.50 per year. Single Number 15 cents.

Perhaps the most beautiful series of pictures ever presented of the Rocky mountains will be found in a collection of fourteen original paintings, executed by Thomas Moran for the May Cosmopolitan. To those who have been in the Rockies, this issue of The Cosmopolitan will be a souvenir worthy of preservation. This number contains fifty-two original drawings, by Thomas Moran, Oliver Herford, Dan Beard, H. M. Eaton, F. G. Attwood, F. O. Small, F. Lix, J. H. Dolph, and Rosina Emmett Sherwood, besides six reproductions of famous recent works of art, and forty other interesting illustrations—ninety-eight in all. Though the Cosmopolitan sells for but fifteen cents, probably no magazine in the world will present for May so great a number of illustrations specially designed for its pages by famous illustrators. The fiction in this number is by F. Hopkinson Smith, Gustav Kobbe, W. Clark Russell, Edgar W. Nye, and T. C. Crawford.

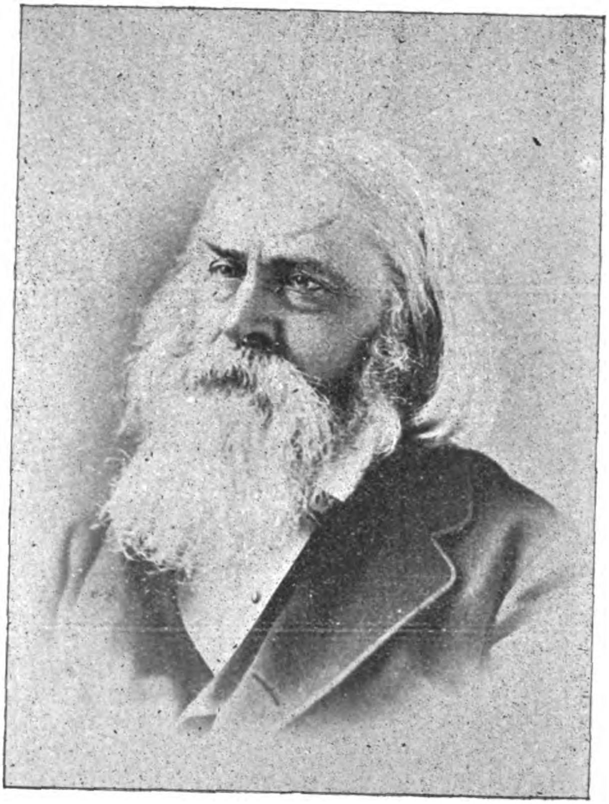
The Ascent of Man. By Henry Drummond, LL.D., F.R.S.E., F.G.S. Fourth Edition. 8vo cloth. 346 pp. Price, $2.00. New York: James Potts & Co., Publishers.

This, to my mind, is the crowning glory of Dr. Drummond's writings. Treating as he does one of the profoundest questions in the history of man, he has told the story of man's Ascent as seen in the light of modern science, in a way that can not fail to impress deeply the most unscientific mind. And though the Author, in his preface, modestly says that the theme is Ascent not Decent, and that the book is a Story not an Argument, yet the Story is so full of sound statements that it has all the force of argument. The book is one we trust will fall into the hands of every lover of scientific research. It is truly scientific and bears no marks of skepticism.
B.F.F.

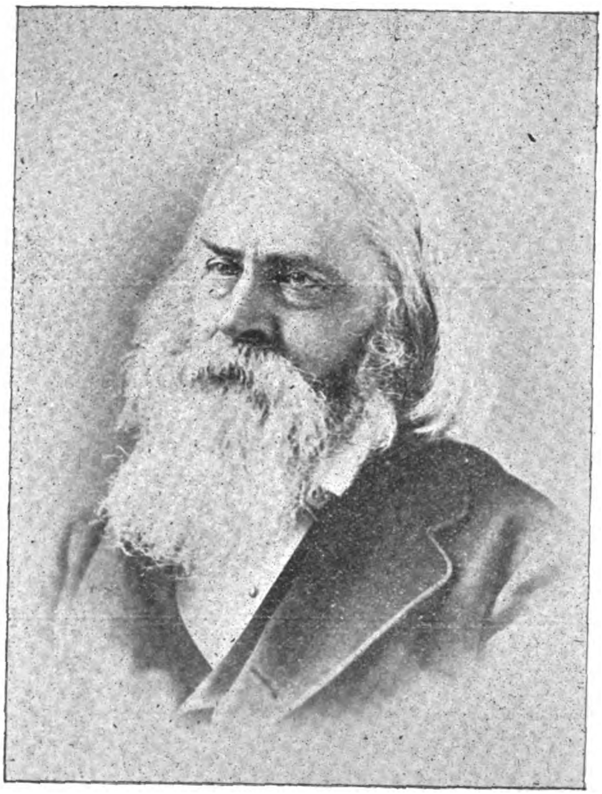
The Basis: A Weekly Journal of Citizenship. Edited by Judge Albion W. Tourgee, Mayville, N. Y., and Published by the Citizens' Publishing Co., Buffalo, N. Y. Price, $1.50 per year. 10 cts. per copy.

The first number of this Journal which has for its object the elevation of the people to higher citizenship and the dissemination of sound principles of government, appeared in April, 1895.

As its editor is one of the leading writers, and one of the ablest jurists in this country, and as the objects of the Journal are noble and unselfish, its success is written on every page. It deals with all questions relating to government and society. Its editor is a fearless and uncompromising advocate of common justice and equal rights. The Basis should be in every home in the land.
B. F. F.



BENJAMIN PEIRCE.



BENJAMIN PEIRCE.

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BIOGRAPHY.

BENJAMIN PEIRCE.

By F. P. MATZ, M Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

BENJAMIN PEIRCE was born at Salem, Massachusetts, April 4, 1809, and died at Cambridge, Massachusetts, October 6, 1880. He entered Harvard College, at the age of sixteen; and, at the age of twenty, he was graduated from the same College, with highest honors. He devoted himself principally to the study of Mathematics. This *favorite* study of his was pursued far beyond the limits of the curriculum of mathematical studies prescribed by the authorities of Harvard College, at that time.

As an under-graduate student, young Peirce was instructed by Nathaniel Bowditch, who soon perceived the innate mathematical genius of his pupil. Bowditch proudly predicted the future greatness of the young man. Not only did Bowditch give him valuable instruction in geometry and analytics, but also acted as his *mathematical adviser*—carefully directing him in the development of his mathematical talents and scientific powers. The lectures on higher mathematics delivered by Francis Grund he was enabled to attend, by reason of his preparation beyond the limit of the under-graduate course in mathematics. When Dr. Bowditch was publishing his translation and commentary of the *Mekanique Celeste* of Laplace, young Peirce assisted in reading the proof-sheets. This critical reading of that great work of Laplace was to him an education in itself, and may have been the prime cause that not a small part of Peirce's subsequent mathematical and scientific work was done in the great field of analytical mechanics:

In the class-room, he frequently gave original demonstrations which proved to be more direct and scientific than those given in the text-books of that day. On graduating, he went to Northampton, Massachusetts, as a teacher in Mr. Bancroft's School. As *tutor*, he returned to Harvard College, in 1831. Since Professor Farrar spent the next year in Europe, tutor Peirce was left at the *head* of the Department of Mathematics in Harvard College: and, on account of the physical inability of Professor Farrar to resume teaching, Peirce *continued* to fill his place. In fact, Peirce held this position, advancing step by step, until the time of his death. His position, in 1842, was christened "The Perkins Professorship of Mathematics and Astronomy." In the history of mathematical teaching at Harvard College, the year 1833 marks an important epoch; as it was then that Benjamin Peirce became the *professor* of Mathematics and Natural Philosophy in that institution of learning.

Professor Peirce was married in July, 1833. At the time of his death, there were living his wife, three sons, and a daughter. His eldest son, James M. Peirce, is University professor of mathematics in Harvard; Charles S Peirce is a professor in the Johns Hopkins University; and H. H. D. Peirce is connected with the firm of Herter Brothers, New York City.

It has been said that a mere boy detected an error in Bowditch's solution of a problem. "Bring me the boy who corrects my mathematics," said Bowditch. Master Benjamin Peirce was the boy who had done the correcting; and thirty years later, this same Benjamin Peirce dedicated one of his great mathematical works "To the cherished and revered memory of my master in science, Nathaniel Bowditch, *The Father of American Geometry*." This same title was bestowed upon Peirce, by foreign mathematicians. Sir Wm. Thomson (Lord Kelvin), in an address before the British Association, referred to Benjamin Peirce as "*The Founder of High Mathematics in America*"; and on a similar occasion, the late Professor Cayley referred to him as "*The Father of American Mathematics*." The name of Benjamin Peirce is that of an *American mathematician*, whom no one need hesitate to rank with the names of Pythagoras, Leibnitz, Newton, Legendre, John Bernoulli, Wallis, Abel, Laplace, Lagrange, and Euler. Through the united efforts of the late Professor Wm. Chauvenet (Yale's ablest mathematician and astronomer) and Benjamin Peirce —not to speak of their worthy successors, was effected the general adoption of the *ratio system* in American works on trigonometry.

In the reforms incident to the *New Education*, Harvard has always taken a prominent part and Benjamin Peirce was *an enthusiastic advocate* of the elective system with respect to collegiate studies. As a branch of Harvard College, there was opened, in 1842, the Lawrence Scientific School; and in this school, Professor Peirce gave instruction in higher Mathematics including analytical and celestial mechanics. Such advanced courses of mathematics, as *he* offered to students, in 1848, had never before been offered to American students by any other professor in any other American college. The second American educational institution which offered equally advanced courses of mathematics, is the Johns Hopkins University; and these courses were arranged by that *English master*, who gave a fresh and powerful impulse to mathematical study and teaching in America—*Professor J. J. Sylvester.*

The preparation of mathematical text-books was begun by Professor Peirce, immediately on beginning his career as teacher of Mathematics in Harvard College. In 1835 appeared his *Elementary Treatise on Plane Trigonometry*; in 1836, his *Elementary Treatise on Spherical Trigonometry* together with his *Elementary Treatise on Sound*; in 1837, his *Elementary Treatise on Plane and Solid Geometry* together with his *Elementary Treatise on Algebra*; during the period 1841-46, he wrote and published in *two* volumes his *Elementary Treatise on Curves, Functions, and Forces*; and in 1855, he published his *Analytical Mechanics*. Subsequently was published his memoir on *Linear Associative Algebra*; and this memoir, according to Professor James Mills Peirce, he regarded as his great work. All of his works are models of conciseness, perspicuity, and elegance; and they all evince extraordinary originality and genius.

In 1867, Professor Peirce was made the Superintendent of the United States Coast Survey; and he held that position for seven years. He had been consulting astronomer to the *American Ephemeris and Nautical Almanac*, since 1849; and for many years, he *directed* the theoretical part of the work. In 1855, Professor Peirce was one of the men intrusted with the organization of the Dudley Observatory. For many years *before* and *after* he took charge of the United States Coast Survey, he was frequently consulted with respect to the work in that office. He received the degree of *Doctor of Laws* from the University of North Carolina, in 1847, and also from Harvard University in 1867. He was elected an Associate of the Royal Astronomical Society of London in 1849 and a member of the Royal Society of London in 1852. He was elected president of the American Association for the Advancement of Science, in 1853 (the fifth year of its existence); and he was one of the *original* members of the Royal Societies of Edinburg, and Gottingen; Honorary Fellow of the Imperial University of St. Vladimir, at Kiev;etc.

Professor Peirce's conception of the American Social Science Association was that it should be a *university for the people*,—combining those who can contribute any thing original in social science into a temporary academical senate, to meet for some weeks in a given place and debate questions with each other, as well as to give out information for the public. In this line of thought he favored, also, the establishment of the Concord School of Philosophy, to do a similar work in the speculative studies; and he lived to see the partial realization of what he foresaw in this instance. In a Mathematical Society over which he presided for some years, each member would bring something novel in his own particular branch of study; and in the discussion which followed, it would almost invariably appear that Professor Peirce had, while the paper was being read, pushed out the author's methods to far wider results than the author had dreamed possible. The same power of extending rapidly in his own mind novel mathematical researches was exhibited at the sessions of every scientific body at which he chanced to be present. What was quite as admirable was the way in which he did it, giving the credit of the thought always to the author of the essay under discussion. His pupils thus frequently received credit for what was in reality far beyond their attainment. He robbed himself of fame in two ways: by giving the credit of his discoveries to those who had merely suggested the line of thought, and by neglecting to write out and publish that which he had himself thought out.

In physical astronomy, perhaps, his greatest works were in connection with the planetary theory, his analysis of the Saturnian system, his researches regarding the lunar theory, and the *profound criticism* of the discovery of Neptune following the investigations of Adams and Leverrier. At the time of the publication of his "*System of Analytical Mechanics*," Professor Peirce announced that the volume would be followed by three others, entitled respectively: "*Celestial Mechanics*," "*Potential Physics*," and "*Analytical Morphology*." These three volumes were never published.

Professor Peirce, in a paper read before the American Association for the Advancement of Science, in 1849, showed in the vegetable world the demonstrable presence of an intellectual plan—showed that phyllotaxis (the science of the relative position of leaves) involved an algebraic idea; and this algebraic ida was subsequently shown to be the solution of a physical problem.

The higher mathematical labors of so eminent a geometer must lie beyond the course of general recognition. Among the things which give him a just claim to this title, may be mentioned: his discussion of the motions of two pendulums attached to a horizontal cord; of the motions of a top; of the fluidity and tides of Saturn's rings; of the forms of fluids enclosed in extensible sacs; of the motions of a sling; of the orbits of Uranus, Neptune, and the comet of 1843; of the criteria for rejecting doubtful observations; of a new form of binary arithmetic, of *systems* of linear and associative algebra; of various mechanical games, puzzles, etc.; of various problems in geodesy; of the lunar tables; of the occultations of the Pleiades; etc. He adapted the epicycles of Hipparchus to the analytical forms of modern science; and he, also, solved by a system of co-ordinates of his own devising, several problems concerning the involutes and evolutes of curves, which would probably have proved impregnable by any other method of mathematical approach.

None of Professor Peirce's labors lie farther above the ordinary reach of thought than his little lithographed volume on Linear and Associative Algebra. In this he discusses the nature of mathematical methods, and the characteristics which are necessary to give novelty and unity to a calculus. Then he passes to a description of seventy or eighty different kinds of simple calculus. Almost no comment is given; but the mathematical reader discovers, as he proceeds, that only *three* species of calculus, having each a unity in itself, have been hitherto used to any great extent,—namely, *ordinary algebra*, *differentials*, and *quaternions*. Think of it; what a wonderful volume of prophecy that is which describes seventy or eighty species of algebra, any one of which would require generation after generation of ordinary mathematicians to develop!

On both sides of the Atlantic, Professor Peirce as an author, was highly esteemed. His work on analytical mechanics was, at the time of its publication, regarded even in Germany, as the *best* of its kind. As a lecturer, Professor Peirce was highly esteemed in both scientific and popular circles. It is related that in 1843, by a series of popular lectures on astronomy, he so excited the public interest that the necessary funds were immediately supplied, for erecting an astronomical observatory at Harvard College. A remarkable series of lectures on "*Ideality in Science*," delivered by him in 1879 before the Lowell Institute in Boston, attracted the general attention of American thinkers, on account of the thoughtful consideration of the vexed question of science and religion.

Professor Peirce was a transcendentalist in mathematics, as Agassiz was in zoology; and a certain subtile tie of affinity connected these two great men, however unlike they were in their special genius. Alike, also, they were in their enthusiasm which neither the piercing scepticism of Cambridge could wither, nor declining years chill with the frost of age. The thing he distrusted was routine and fanatical method, whether new or old; for thought, salient, vital, co-operative thought, in novel or in ancient aspects, he had nothing but respect and furtherance. Few men could suggest more while saying so little, or stimulate so much while communicating next to nothing that was tangible and comprehensible. The young man who would learn the true meaning of *apprehension* as distinct from *comprehension*, should have heard the professor lecture, after reciting to him. He was always willing to be esteemed for less than he had really accomplished; and he could join most heartily in the praise of others who even owed their impulse to him. *Modest* and *magnanimous*, but not unobservant, his ambition for personal distinction was early and easily satisfied; and he thus rid himself of what is to most men a perturbing, and too often an ignoble, element of discomfort.

Professor Peirce habitually ascribed to his listener a power of assimilation which the listener rarely possessed. He assumed his readers could follow wherever he led; and this made his lectures hard to follow, his books brief, difficult, and comprehensive. When, however, his listeners were students who had previously attained some skill as mathematicians and who had been trained in his own methods, the resulting work would be of *the highest order of excellence.* He was personally magnetic in his presence. His pupils loved and revered him; and to the young man, he always lent a helping hand in science. He inspired in them a love of truth for its own sake.

His own faith in Christianity had the simplicity of a child's; and whatever radiance could emanate from a character which combined the greatest intellectual attainment with the highest moral worth, that radiance cast its light upon those who were in his presence. "*Every portion of the material universe,*" writes Professor Peirce, "*is pervaded by the same laws of mechanical action which are incorporated into the very constitution of the human mind.*" To him, then, the universe was made for the instruction of man. With this belief he approached the study of natural phenomena not in the spirit of a critic, but reverently in the mood of a sympathizing reader; and the lesson he reads is: "*There is but one God, and science is the knowledge of Him.*" In his lectures and teaching he showed, as he always felt with adoring awe, that the mathematician enters (as none else can) into the intimate thought of God, sees things precisely as they are seen by the Infinite Mind, holds the scales and compasses with which the Eternal Wisdom built the earth and meted out the heavens. This consciousness had pervaded his whole scientific life. It was active in his early youth, as his coevals well remember; it gathered strength with his years; and it struck the ever recurring key-note in his latest public utterances.

Benjamin Peirce was a devout, God-fearing man; he was a Christian, in the whole aim, tenor, and habit of his life. To know Professor Peirce was simply to love him, to admire him, and to revere him. Since he was conversant with the phases of scientific infidelity, and by no means unfamiliar with the historic grounds of scepticism, it can not be regarded otherwise than with the profoundest significance, that a *mind* second to none in keen intuition, in aesthetic sensibility, in imaginative ferver, and in the capacity of close and cogent reasoning, *maintained* through life an unshaken belief and trust in the power, providence, and love of God, as beheld in his works, and as incarnate in our Lord and Savior. In one of his lectures on *Ideality in Science*, he said: "Judge the tree by its fruit." Is this magnificent display of ideality a human delusion? Or is it a divine record? The heavens and the earth have spoken to declare the glory of God. It is not a tale told by an idiot, signifying nothing. It is the poem of an infinite imagination, signifying immortality."

In May, 1880, Professor Peirce began to pass under the shadow of the cloud of his last illness. For some weeks there was little serious fear that it was a shadow not destined to lift. He was first confined to his chamber, on the 25th of June, 1880; and from that time, his slowly failing condition was hardly relieved even by any deceptive appearances of improvement. He died on the morning of Wednesday, October 6, 1880. Distinguished throughout his life by his freedom from the usual abhorrence of death, which he never permitted himself either to mourn when it came to others, or to dread for himself, he kept this characteristic temper to the end, through all the sad changes of his trying illness; and, two days before he ceased to breathe, it struggled into utterance in a few faintly-whispered words, which expressed and earnestly inculcated a cheerful and complete acceptance of the will of God with regard to him.

The funeral took place on Saturday, October 9, 1880, at Appleton Chapel, and was the occasion of an impressive gathering of people of great and various mark. The attendance included a very full representation of the various faculties and governing boards of the University; a large deputation of officers of the United States Coast and Geodetic Survey, headed by the superintendent and the chief assistant; delegations of eminent professors from Yale College and the Johns Hopkins University; many members of the class of 1829; and a great number of other friends of the deceased.

The pall-bearers were: President Charles W. Eliot; Ex-President Thomas Hill, Pastor of the First Parish Church, Portland, Maine; Capt. C. P. Patterson, Superintendent of the United States Coast Survey; Professor J. J. Sylvester, of the Johns Hopkins University; Hon. J. Ingersoll Bowditch; Professor Simon Newcomb, Superintendent of the American *Ephemeris and Nautical Almanac*; Dr. Oliver Wendell Holmes; Professor Joseph Lovering; and Dr. Morrill Wyman. A beautiful and simple service was conducted by the Rev. A. P. Peabody and the Rev. James Freeman Clarke.

In the career of Professor Benjamin Peirce, America has nothing to regret, but that it is now closed; while the American people have much to learn from his long, useful, and honorable life.

REMARKS ON SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., *Professor of Mathematics*, University of Michigan, Ann Arbor, Michigan.

[Continued from May Number.]

Since $a.bbc=acb$ and $bc, ab=abc$ it follows that the result is not always independent of the order in which we perform the operations indicated by two substitutions. In the equation $ab.bc=acb$ we call ab and bc the *factors* and acb the *product* and the process is called *multiplication of substitutions*. The above example shows that the law that the product is independent of the order of the factors does not hold true with respect to the multiplication of substitutions.

The number of substitution groups increases very rapidly as the number of letters increases. During the last few years the work of making complete lists of such groups has been carried through ten letters* but no formula has yet been published by means of which the number of such groups can readily be determined for any number of letters.

If an expression involving a given number of letters is unchanged by applying all the substitutions of a group of the same number of letters to it but is changed by applying any other substitution of the same or a lower number

*The number of groups of ten letters exceed one thousand. Complete lists are found in the Quartly Journal of Mathematics as follows: Cayley: Substitution groups for two, three, four, five, six, seven, and eight letters, vol. 25, pp. 71-88, 137-155. Cole: List of substitution groups of nine letters, vol. 26, pp. 372-388. Cole. List of transitive substitution groups of ten and eleven letters, vol. 27, pp. 39-50. Miller: Intransitive groups of ten letters, vol. 27, pp 99-118.

A few errors and omissions with respect to these lists have been noted in late numbers of the Bulletin of the American Mathematical Society. The lists are complete in the sense that an effort is made to give all the possible groups of the given number of letters and fairly accurate results have been attained.

of letters, then the expression is said to belong to the given group. We may take, for example, the expression with four letters

$$ac+bd.$$

We see that no matter what values a, b, c, and d may have this expression cannot change its value for any of the substitutions in

$$\begin{array}{llll} 1 & ab.cd & abcd & ac \\ & ac.bd & adcb & bd \\ & ad.bc & & \end{array}$$

but that it changes its value, in general, for any other substitution of four or a lower number of letters; hence we say the given expression belongs to this group, and, conversely, that the group belongs to this expression. From this it can be seen that substitution groups furnish a means by which we may classify such algebraic expressions and thus study the common properties once for all. It has been proved that every integral expression belongs to some group and that an infinite number of such expressions belong to each group. By studying a group we therefore study some properties common to an infinite number of algebraic expressions and from this it follows that the study of substitution groups is a matter of economy in case familiarity with a large number of expressions is to be attained.

The groups of two and three letters are so simple that they are frequently employed without any explanation of their connection with an extensive science. For example, when a factor of an expression belonging to one of these groups can be found by inspection and belongs to the same group as the expression the form of the other factor is often obtained from the fact that it must belong to the same group. This is explained by employing simple properties of these groups in place of the groups themselves. Even in these cases a knowledge of the theory of substitution groups would contribute much to the clear understanding of the matter on the part of the student and in the more complex cases such a knowledge becomes almost indispensible if a comprehensive knowledge is to be attained. In the factoring known as the solution of equations, substitution groups have since the time of Galois played the most prominent part—serving not only to give a comprehensive view of the entire field but also to extent the knowledge with respect to it.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton), Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the May Number.]

PROPOSITION XIX. *Let there be any triangle AHD* (fig.18) *right angled at H. Then in AD produced the portion DC is assumed equal to this AD; and the perpendicular CB is let fall to AH produced. I say hence will be established the hypothesis of right angle, or obtuse angle, or acute angle, according as the portion HB is equal to, or greater, or less than this AH.*

PROOF. For the join DB will be (Eu.I.4, and P. X of this) either equal to, or greater, or less than AD, or DC, according as the portion HB is equal to, or greater, or less than AH. And first indeed let HB be equal to AH, so that therefore the join DB may be equal to AD, or DC.

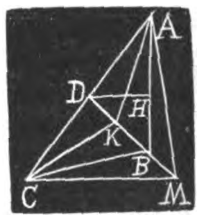


Fig. 18.

It follows that the circumference of the circle, which is described with the center D, and radius DB, will go through the points A, and C. Therefore the angle ABC, which is assumed right, is in this semicircle, whose diameter is AC. Wherefore (from the preceding proposition) is established the hypothesis of right angle. Quod erat primo loco demonstrandum.

Secondly let BH be greater than AH, so that therefore the join DB is greater than AD, or DC. It follows that the circumference of the circle, which is described with center D, and radius DA, or DC, will meet DB in some intermediate point K. Therefore, AK, and CK being joined, the angle AKC will be obtuse, because greater (Eu. I. 21) than the angle ABC, which is assumed right. Wherefore (from the preceding proposition) is established the hypothesis of obtuse angle. Quod erat secundo loco demonstrandum.

Thirdly let BH be less than AH, so that therefore the join DB is less than AD, or DC. It follows that the circumference of the circle, which is described with center D, and radius DA, or DC, will meet in some point M this DB produced outwardly. Therefore AM, and CM being joined, the angle AMC will be acute, because less (Eu. I. 21) than the angle ABC, which is assumed right.

Therefore (from the preceding proposition) is established the hypothesis of acute angle. Quod erat tertio loco demonstrandum. Itaque constant omnia proposito.

[To be continued.]

THE REALIZATION OF IMAGINARY POINTS.

By WARREN HOLDEN, Professor of Mathematics, Girard College, Philadelphia, Pennsylvania.

In Salmon's Conic Sections, Art. 82, the author says of an imaginary point: "It is a purely analytical conception, which we do not attempt to represent geometrically," * * * * "but attention to these imaginary points is necessary," * * * * "we shall meet with many cases in which the line joining two imaginary points is real."

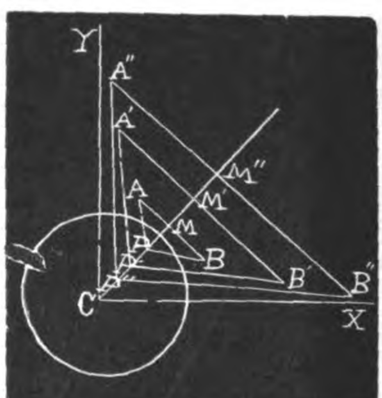
It is here proposed to trace some of these imaginary points, in the hope of finding them upon the real line which joins them.

Take the points of contact of tangents to a circle from a given point. The tangents are said to be real when the point from which they are drawn is without the circle, coincident when the point is on the circle, and imaginary when the point is within the circle. It is these last tangents and their points of contact, which it is proposed to find. These last points will be found, if any where, upon the polar of the given point, which is the "real line joining the imaginary points."

The co-ordinate of contact of tangent to a circle from any point $x'y'$, are $x''=\dfrac{R^2x'\pm Ry'\sqrt{x'^2+y'^2-R^2}}{x'^2+y'^2}$, $y''=\dfrac{R^2y'\mp Rx'\sqrt{x'^2+y'^2-R^2}}{x'^2+y'^2}$.

It will be shown that these formulas may be so interpreted as to determine the points of contact of tangents, from a point *within* the circle, to a curve which may be regarded as a mere development of the circle, and that the line joining these points is the polar of the given point.

When $x'y'$ reaches the circumference the expression under the radical sign becomes $x'^2+y'^2-R^2=0$ or $x'^2+y'^2=R^2$, which may be written $R^2-(x'^2+y'^2)$. *Proceeding now with the application of the formulas we obtain the following results:



Let C be the centre of a circle, CX, CY the axes, the radius equal to 4, and the distances of the points P, P', P'' from the centre respectively 3, 2, 1. Then for P or (x^1, y^1) we have

$$x''=\frac{16\times2.12\pm4\times2.12\sqrt{16-9}}{9}$$

$=6.25$ or 1.28: $y''=1.28$ or 6.25. These determine the points A, B. The line joining these points is the polar of P, as proved by $CM=\dfrac{R^2}{CP}=\dfrac{16}{3}=5\frac{1}{3}$. For P' we have $x''=\dfrac{16\times1.4\pm4\times1.41\sqrt{16-4}}{4}$

*By substituting the sign difference in place of the sign minus in the expression under the radical, the same result is reached.

$=10.44$ or 0.75, which give the points A', B'. Joining $A'B'$ we have the polar of P' verified by $CM' = \frac{R^2}{CP'} = \frac{16}{2} = 8$. For P'' $(x'y')$ we have $x' = 16 \times 0.707 \pm 4 \times 0.707\sqrt{16-1} = 22.225$ or 0.369. These determine the points A'', B''. The line joining these points $A''B''$ is the polar of P'', as proved by $CM'' = \frac{R^2}{CP''} = \frac{16}{1} = 16$.

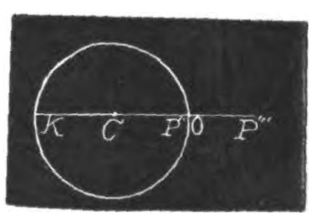
The locus of $x''y''$, while P or $x'y'$ moves along the same radius, is an equilateral hyperbola, concentric with the given circle and having its vertex at the circumference.

The proof is as follows: The co-ordinates of contact of tangents to a hyperbola, when equilateral and its semi-axes each equal to R, are $x'' = \frac{R^2x' \mp Ry'\sqrt{R^2-(x'^2-y'^2)}}{x'^2-y'^2}$, $y'' = \frac{R^2y' \pm Rx'\sqrt{R^2-(x^2-y'^2)}}{x^2-y'^2}$. CM is now the axis of X's, and since $P(x'y')$ moves along this line, $y'=0$. The formulas give for P, $x'=3$, $x''=5\frac{1}{3}$ or CM. $y''=\pm 3.52$ or MA and MB, which determine the points A, B. For $P'=x'=2$, $x''=8$ or CM', $y''=\pm 6.92$ or $M'A'$ and $M'B'$. For P'', $x'=1$, $x''=16$, or CM'', $y''=\pm 15.48$ or $M''A''$ and $M''B''$, all the same points as by the first formulas. Thus it appears that the *imaginary* tangents to a circle are *real* tangents to a corresponding hyperbola.

An attempt will now be made to justify the above results without recourse to the disputable expedient of changing the signs of the expression under the radical.

It is now understood that the difference between positive and negative quantities is merely the difference of counting in opposite directions, both equally real. Guided by this hint, if we let (a^2) stand for the expression under the radical sign, and consider that $(-a^2)$ is composed of factors, one of which is affected with the sign $(-)$, then, when $\sqrt{a^2}(=a\sqrt{+1})$ becomes $\sqrt{-a^2}(=a\sqrt{-1})$, we know that a factor has changed the direction in which it was before estimated, to the opposite direction. By examination we discover where and how the change took place.

In the present case it is the reversal of direction in counting from the the point $x'y'$ to the circumference. This may be seen by factoring the expression under the radical. Taking the point P''' $x'y'$ when without the circle, $x'^2+y'^2-R^2 = P'''C^2 - OC^2 = (P'''C + OC)(P'''C - OC) = P'''K \times P'''O$, both factors measured in the same direction. When we reach P'' *within* the circle, we have $P''K \times P''O$, in which $P''O$ is measured in the opposite direction and of course takes the sign $(-)$, which explains the change of sign in the radical expression $(\sqrt{-a^2})$. In other words, the instant that $x'y'$ crosses the circumference, $\sqrt{-1}$ appears. Call it then the sign of that crossing and nothing else, and at once the imaginary is divested of its badge of licensed irresponsibility



and falls into the ranks of orderly realities, which alone properly constitute the exact sciences. Instead of being "an expression for an impossible operation" the $\sqrt{-1}$, in its present situation at least, calls for no operation whatever, its origin being accounted for, and its office defined just as it stands—a finger-post where before it stood an impassable barrier.

The $\sqrt{-1}$ takes its rise as a residual factor or coefficient of the term whose root has been extracted. Like the corresponding factor, $\sqrt{+1}$ it has no power to either increase or diminish the *numerical* value of the term. Why then treat it as having any numerical significance? Why not call it the co-efficient of *direction*? Whether the direction be *opposite* or *perpendicular*, say that while the symbol denotes *change* of direction the *particular* direction must in each instance, be determined by its own conditions. Although by convention (—) indicates direction opposite to (+), that will not prevent $\sqrt{-1}$ from indicating the same when the conditions of its appearance distinctly point that way, provided this use of the symbol involves no inconsistency with its other uses.

The conditions in the present example may be restated thus: Of the elements which enter into the radical expression, viz. R and $x'y'$, the latter is referred to the limits of the former, that is to the circumference and the centre. While $x'y'$ is without the circle, both limits lie in the same direction from it. After $x'y'$ enters the circle, the limits (centre and circumference) lie in opposite directions from it. And this change is signalized by the appearance of $\sqrt{-1}$.

If C be a fixed point and $x'y'$ a moving point, approaching C on a right line from an infinite distance, then by the application of the first formulas above, the tangents will generate, by a *continuous movement*, first the semi-circumference and then the hyperbola: showing the hyperbola to be an inverse continuation of the circle, or the circle turned inside out.

When $x'y'$ reaches the limit C the tangents find their true limits in the asymtotes, and coincide with the axes. The momentary coincidence of the tangents at the other limit—the circumference—only marks the transition from one phase of their work to the other.

The tangents, moving in obedience to a uniform law, would naturally be expected, after generating the semi-circumference, not to wander off into "untraceable labyrinths," but to follow some plain path, especially as the line joining their (imaginary) points of contact, may, at any stage, be definitely located.

OUTLINE OF INVESTIGATION FOR ASYMPTOTES.

By E. S. LOOMIS, A. M., Ph. D., Professor of Mathematics in Baldwin University, Berea, Ohio.

I. Methods.

A. By Inspection.

1. If when $x \doteq 0$, $y \doteq \infty$, $x=0$ is an asymptote.
2. " " $x \doteq 0$, $y \doteq a$, or $\sqrt{-a}$, $x=0$ is not an asymptote.
3. " " $x \doteq \pm a$, $y \doteq \pm \infty$, $x=\pm a$ is an asymptote.
4. " " $x \doteq \pm a$, $y \doteq \pm a$, or $\pm\sqrt{-a}$, $x=\pm a$ is not an asymptote.
5. Treat y in some manner as x.

Note 1. To be universally true, the equation must be solved for either y or x.

Note 2. The finite quantity is seen in the curve.

B. By direct Investigation.

1. Through Intercepts.

(a). If X and Y, either or both, are finite, there is an asymptote.

(b). If X and Y are both infinite, no asymptotes.

(c). To find the equation of the asymptote, substitute the intercepts of X and Y for a and b in $\frac{x}{a}+\frac{y}{b}=1$.

Note. Under (a), if need be, find the limit of $\frac{y}{x}$.

2. By writing for y, $kx+r$, in the curve, expanding, arranging according to the descending powers of x, writing the coefficients of the two highest powers of x equal to 0, from which find the values of k and r, which values substituted in last two terms of arranged equation just found give the equation of an asymptote.

3. By solving the equation for y and developing by Maclaurin's formula, etc.

II. An example of application and illustration. I shall take the cissoid (of Diocles), because I have never seen it worked, either by text or student, except by the method of inspection. Of course by inspection, by 3rd under A above it is instantly seen that $x=2a$ is an asymptote. I have had students declare that it could not be solved by direct investigation. It comes directly under (a) of B above. First, as in Analytic Geometry, investigate for limits of the curve. We discover it is limited by $2a$ to the right. From $y^2=\frac{x^3}{2a-x}$, $\frac{dx}{dy}=\frac{2y(2a-x)}{3x^2+y^2}$ second in $X=x-y\frac{dx}{dy}$, sub. value of y and $\frac{dx}{dy}$, gives

$$X=\frac{ax^3}{3ax^2-x^3}=\frac{ax}{3a-x}=\frac{2a^2}{a}=2a, \text{ since } x=2a \text{ at limit.}$$

Concluded on page 204.

ARITHMETIC.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTION OF PROBLEMS.

47. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, in New Windsor College, New Windsor, Maryland.

Mr. Merchant sells 20% above cost, with weights and measures $12\frac{1}{2}$% "short," allows a discount of \$5 on every bill of \$50, and loses 5% of his sales as "bad debts." Find his *rate per cent.* of net profit, or net loss; one cent in every dollar of sales proves counterfeit, and collection-charges are $2\frac{1}{2}$%.

I. Solution by H. W. DRAUGHON, Olio, Mississippi, and the PROPOSER.

Let $\$C$=the cost of the merchandise sold; then $\frac{120}{100}$ of $\frac{8}{7}$ of $\$C=\frac{48}{35}$ of $\$C$=the amount of the merchandise sold. As per the problem, the *aggregate* of deductions to be made from the *amount of the sales* is $18\frac{1}{2}$%; that is, the *net* amount of the sales is $\frac{978}{875}$ of $\$C$, and the *net* profit is $\frac{103}{875}$ of $\$C$. Hence the required rate per cent. of *net* profit must be $11\frac{27}{35}$.

II. Solution by G. B. M. ZERR, A. M., Principal of High Schools, Staunton, Virginia.

$100\%-12\frac{1}{2}\%=87\frac{1}{2}\%$, what he sells for 120%.

$120\%\div87\frac{1}{2}=137\frac{1}{7}\%$, what he gets for 100%.

\$5 on \$50=10%.

$10\%+5\%+1\%+2\frac{1}{2}\%=18\frac{1}{2}\%$, what he loses.

$137\frac{1}{7}\times18\frac{1}{2}=25\frac{13}{35}\%$.

$137\frac{1}{7}-25\frac{13}{35}=111\frac{27}{35}\%$. $\therefore$ he gains $11\frac{27}{35}\%$.

Also solved by *P. S. BERG.*

PROBLEMS.

52. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College New Windsor, Maryland.

By selling a horse for H=\$150 cash, I gain $p=20\%$. At what price should I sell the horse and wait $d=90$ days, money worth $m=6\%$, in order to gain $q=25\%$?

53. Proposed by P. S. BERG, Apple Creek, Ohio.

\$500 Wooster, O., Sept. 2nd, 1886.

One year after date we, or either of us promise to pay to the order of J. M. W. Five Hundred Dollars for value received with 7% annual interest from date.

J. C.
M. C.

Endorsed May 13, 1893, \$75.00
" Sept. 1, 1894, \$300.00.

What was due April 1st, 1895?

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTION OF PROBLEM.

43. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

Four men, A, B, C, and D, start from the same place, the traveling rates of A and C are equal, and the traveling rates of B and D were as 17 to 18, respectively; B could travel one mile in 7 minutes and 12 seconds. A traveled *due* west a certain distance, B traveled *due* north the cube of A's distance *plus* his distanc; C traveled *due* east a certain distance, and D traveled *due* south the cube of C's distance *plus* his distance: They all change directions, and A traveled *due* north a certain distance, B traveled *due* east the 5th power of A's distance north; C traveled *due* south a certain distance, and D traveled *due* west the 5th power of C's distance south,—when it was found that the sum of the north and south distances traveled by B and D was 351090 feet, and the sum of the distances that B and D traveled east and west was 5939200000 feet, and that the product of the distances that A and C traveled east and west *plus* the square of the difference of these distances, *plus one* was 3901: and that the product of the distances that A and C traveled north and south *plus* the square of the difference *squared*, plus the product multiplied by the square of the difference, was 494100.00 [equal to the following new formulas: $(nn+d^2+1)=3901$, and $\{(nn+d^2)^2+(nn\times d^2)\}=49410000$]. How far on a line is each party from the starting place, and how long did it require for B and D each to make the entire trip from starting place to the end?

Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let x=distance A travels due west
y= " C " " east
z= " A " " north
u= " C " " south.

Then x^3+x=distance B travels due north
y^3+y= " D " " south
z^5= " B " " east
u^5= " D " " west

Also $x^3+y^3+x+y=351090$......(1)
$z^5+u^5=5929200000$......(2)
$xy+(x-y)^2+1=3901$......(3)
$\{zu+(z-u)^2\}^2+zu(z-u)^2=49410000$......(4)

From (3), $x^2-xy+y^2+1=3901$......(5)
From (1), $(x+y)(x^2-xy+y^2+1)=351090$......(6)
(5) in (6) gives, $x+y=90$......(7)
(7) in (1) gives, $x^3+y^3=351000$......(8)
(7) cubed minus (8) gives, $3xy(x+y)=378000$......(9)
(7) in (9) gives, $xy=1400$......(10)

From (7) and (10), $x=70$ or 20, $y=20$ or 70
From (2), $(z+u)(z^4-z^3u+z^2u^2-zu^3+u^4)=5929200000$......(11)
From (4), $z^4-z^3u+z^2u^2-zu^3+u^4=49410000$......(12)

(12) in (11) gives, $z+u=120\ldots\ldots(13)$

$[\{(13)\}^5-(2)]\div5$ gives, $zu(z^3+2z^2u+2zu^2+u^3)=3790800000\ldots\ldots(14)$

$\{(13)\}^3$ in (14) gives $zu(1728000-z^2u-zu^2)=3790800000\ldots\ldots(15)$

(13) in (15), after reduction, gives, $z^2u^2-14400zu=-31590000$

$\therefore zu=11700$ or $2700\ldots\ldots(16)$.

From (13) and (16) we get, $z-u=\pm60$ or $\pm180\sqrt{-1}$.

$\therefore z=90$ or 30 or $30(2\pm3\sqrt{-1})$,

$u=30$ or 90 or $30(2\mp3\sqrt{-1})$.

A is distant from the starting point either of the following:—

$\sqrt{90^2+70^2}=114.017+$feet, $\sqrt{90^2+20^2}=92.196$ feet.

$\sqrt{30^2+70^2}=76.157$ feet, $\sqrt{30^2+20^2}=36.055$ feet.

C is distant from the starting point either of the following:—

36.055 feet, 76.157 feet, 92.196 feet, 114.017 feet.

B is distant from the starting point either of the following:—

$\sqrt{(70^3+70)^2+(90^5)^2}=5904900009.965$ feet,

$\sqrt{(20^3+20)^2+(90^5)^2}=5904900000.005$ feet,

$\sqrt{(70^3+70)^2+(30)^{10}}=24302421.629$ feet,

$\sqrt{(20^3+20)^2+(30)^{10}}=24300001.323$ feet.

D is distant from the starting point either of the following:—

24300001.323 feet, 24302421.629 feet, 5904900000.005 feet, 5904900009.965 feet.

B has traveled either of the following distances:

$70^3+70+90^5=5905243070$ feet $=118417.38447$ miles,

$20^3+20+90^5=5904908020$ feet $=118353.79167$ miles,

$70^3+70=30^5=24643070$ feet $=4667.24811$ miles,

$20^3+20+30^5=24308020$ feet $=4603.79167$ miles.

D has traveled either of the following distances:

4603.79167 miles, 4667.24811 miles, 118353.79167 miles, 118417.38447 miles.

B travels 1 mile in 7 min. 12 sec. $=432$ sec.

D travels one mile in $\frac{17}{18}$ of $432=408$ sec.

It has taken *B* either of the following times:—

$118417.38447\times432=51156310.09$ sec. $=14210$ h. 5 m. 10.09 sec.,

$118353.79167\times432=51128838$ sec. $=14202$ h. 27 m. 18 sec.,

$4667.24811\times432=2016251.18$ sec. $=560$ h. 4 m. 11.18 sec.,

$4603.79167\times432=1988838$ sec. $=552$ h. 27 m. 18 sec.

It has taken *D* either of the following times:—

$4603.79167\times408=1878347$ sec. $=521$ h. 45 m. 47 sec.,

$4667.24811\times408=1904237.22$ sec $=528$ h. 57 m. 17.22 sec.,

$118353.79167\times408=48288347$ sec. $=13413$ h. 25 m. 47 sec.,

$118417.38147\times408=48314292.86$ sec. $=13420$ h. 38 m. 12.86 sec.

The imaginary results have been omitted as impossible.

The above solution was the first received and entitles *Prof. ZERR* to the St. Andrews, Fla., City Lot offered by the *PROPOSER*. Excellent solutions of later date were received from *H. C. WHITAKER*, *M. A. GRUBER*, *H. W. DRAUGHON*, *H. C. WILKES*, *A. H. BELL*, *A. L. FOOTE*, *A. H. HOLMES*, and *P. S. BERG*.

PROBLEMS.

52. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

In how many ways can we arrange 12 friends of the MONTHLY, around a table, so that; (1), the editors may never be together, (2), Matz and Halsted may never be apart; and (3), Zerr and Ellwood may always have Gruber betwixt them?

53. **Proposed by LEONARD E. DICKSON, M.A., Fellow in Mathematics, University of Chicago.**

Can it be proven that the value of the expression

$$\Big\{ 5x - \frac{5x+3}{3} \cdot \frac{5x(5x-1)}{1.2} + \frac{(5x+3)(5x+8)}{4.5} \cdot \frac{5x(5x-1)(5x-2)}{1.2.3} - \frac{(5x+3)(5x+8)(5x+13)}{5.6.7} \cdot$$

$$\frac{5x(5x-1)(5x-2)(5x-3)}{1.2.3.4} + \frac{(5x+3)\ldots(5x+18)}{6.7.8.9} \cdot \frac{5x\ldots\ldots(5x-4)}{1\ldots.5} + \ldots$$

$$+(-1)^{5x-2} \frac{(5x+3)\ldots(30x-12)}{5x \cdot (5x+1)\ldots(10x-3)} \cdot 5x + (-1)^{5x-1} \frac{(5x+3)\ldots(30x-7)}{(5x+1)\ldots(10x-1)} \Big\}$$

is identically zero?

GEOMETRY.

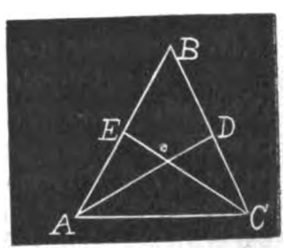
Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEM.

42. **Proposed by G. I. HOPKINS, Instructor in Mathematics and Physics in High School, Manchester, New Hampshire.**

If the bisectors of two angles of a triangle are equal the triangle is isosceles.

II. **Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia; H. C. Whitaker, M. S., M. E., Professor of Mathematics. Manual Training School, Philadelphia, Pennsylvania; WILLIAM HOBBY, a Student in University of Tennessee, Knoxville, Tennessee.**



Let $AC=b$. Then $AD=b\sin C/\sin(C+\frac{1}{2}A)$,

$CE=b\sin A/\sin(A+\frac{1}{2}C)$.

$\therefore \sin C\sin(A+\frac{1}{2}C)=\sin A\sin(C+\frac{1}{2}A)$.

Let $x=\sin\frac{1}{2}A$, $y=\sin\frac{1}{2}C$.

Then $(2xy-2x^3y-y^2+2x^2y^2)\sqrt{1-y^2}$

$$=(2xy-2xy^3-x^2+2x^2y^2)\sqrt{1-x^2}.$$

$\therefore (x^2-y^2)\{x^2+y^2-x^4-x^2y^2-y^4-4xy$
$+4x^3y+4xy^3-4x^3y^3\}=0.$

$\therefore x=y$ and $\angle A=\angle C$,

also, $(x-y)^4-(x-y)^2+xy(1-xy)+xy(1-2xy)^2=0.$

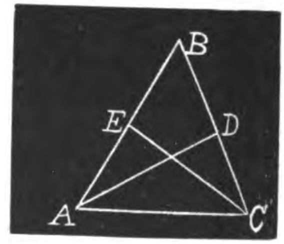
III. **Solution by J. H. GROVE, Howard Payne College, Brownwood, Texas.**

(Solution by Reductio ad Absurdum)

Suppose the $\triangle$ to be scalene and that $AB>BC$.

(1) $AB.AC=BD.DC+AD^2$

(2) $BC.AC=BE.AE+CE^2$



Then (1) $AB.AC-BD.DC=AD^2$ and

(2) $BC.AC-BE.AE=CE^2$. But $AD^2=CE^2$. Hyp.

$\therefore$ $AB.AC-BD.DC=BC.AC-BE.AE$.

But $AB.AC>BC.AC$ (Hyp.), and

$BD.DC<BE.AE$ ($\angle a<\angle c$ Hyp.)

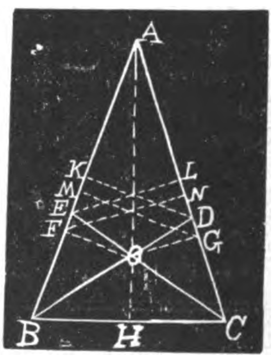
($BD<BE$ and $DC<AE$)

$\therefore$ The conclusion above reached: $AB.AC-BD.DC=BC.AC-BE.AE$ is absurd. It can be true only in case the $\triangle$ is isosceles.

$\therefore$ If the bisectors AD and CE are equal, $AB=BC$. *Q. E. D.*

IV. **Solution by EDW. R. ROBBINS, Master in Mathematics of Lawrenceville School, Lawrrnceville, New Jersey.**

Let ABC be a $\triangle$ of which EC and BD are equal bisectors of base angles. To prove the $\triangle$ is isosceles. Draw third bisector AO. Draw perpendiculars OH, OF, OG: these are equal lines. From E, D, F, G, draw perpendiculars to opposite side.



In $\triangle$s AFO, AOG: $FO=OG$: $AO=AO$ and angles at A are equal.

Therefore these triangles are equal right triangles, and $AF=AG$. In right $\triangle$s AGM and AFN angle at A is common and $AF=AG$. Hence $\triangle$s are equal and $FN=GM$.

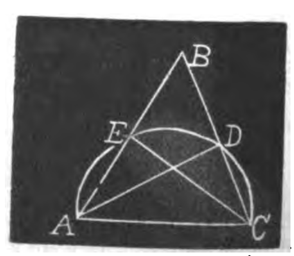
Now the right triangles AEL and ADK are similar respectively to AFN and AGM. But these last $\triangle$'s are equal and hence the $\triangle$'s AEL and ADK are equal. And therefore $AE=AD$.

The triangles ADB and AEC have two sides of the one, AD, DB equal respectively to two sides of the other AE, EC, also the angle at A is common. Hence the $\triangle$s are equal. Therefore $AB=AC$ and original $\triangle$ ABC is isosceles.

Q. E. D.

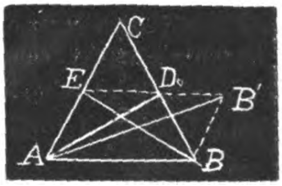
V. **Solution by P. S. BERG, Apple Creek, Ohio.**

Let ABC be a triangle, $AD=CE$. Through the three points A, C and E pass the circumference of a circle. It will also pass through D. For if it meets AD ni P between A and D the arc EA must be greater than PE, since ECA which is equal to DCE is greater than PCE. Also the arc PE is equal to the arc PC, since the angle EAP is equal to the angle PAC. Whence the arc AEP is greater than



EPC, and consequently the chord AP is greater than the chord CE. But by hypothesis $CE=AD$; then AP is greater than AD which is impossible. Hence the circle which passes through A, C, and E cannot cut AD between A and D. In like manner it can be shown that the circle cannot cut AD beyond D. Hence it must pass through D. Hence the angle EAD which is half of A is equal to DCE which is half of C. Therefore the angle A is equal to the angle B and the triangle is isosceles. *Q. E. D.*

VI. Solution by W. W. MOSS, Instructor in Mathematics, Brown University, Providence, Rhode Island.



Let AD and BE be equal bisectors of the $\angle$s CAB and CBA of the $\triangle ABC$. To prove $\triangle ABC$ isosceles. Move the $\triangle DBA$ so that the side AD will coincide with its equal EB, A falling at E, D at B. Then fold $\triangle BAD$ upon AD (EB) as axis till B falls upon the plane at B', $\triangle BAD$ taking position EBB'. Draw AB'. Consider the $\triangle$s AEB' and ABB' $\angle AEB'=\angle AEO+\angle OEB'$ $=\angle AEO+\angle OAB=\angle AEO+\angle OAE=\angle AOB$ $\angle ABB'=\angle ABE$ $+\angle EBB'=\angle OBD+\angle ODB=\angle AOB$. $\therefore$ $\angle AEB'=\angle ABB'$ $\angle CAB$ $+\angle CBA<180^\circ$ and halving $\angle OAB+\angle OBA<90^\circ$. $\therefore$ $\angle AOB=180^\circ$ $-$(angle OAB+angle OBA)$>90^\circ$. $\therefore$ angles AEB' and ABB' are obtuse and equal side EB'=side AB and $AB'=AB'$. $\therefore$ $\triangle AEB'=\triangle ABB'$ having two sides and an opposite angle in one equal to homol. parts in the other, the equal angles being obtuse. $\therefore$ angle EAB=angle EAB'+angle BAB'=angle $AB'B$ +angle $AB'E$=angle $EB'B$=angle DBA.

$\therefore$ $\triangle ABC$ is isosceles. *Q. E. D.*

Solutions of this problem were received from *J. C. CORBIN, WILLIAM PARKISON, F. P. MATZ, O. W. ANTHONY, A. M. HUGHLETT, H. W. DRAUGHON* and *J. F. W. SCHEFFER. Professor SCHEFFER sent in three solutions and Professor GROVE two.*

Note.—An excellent demonstration of this proposition is given on page 44, of *Dr. Halsted's Elementary Synthetic Geometry.*

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

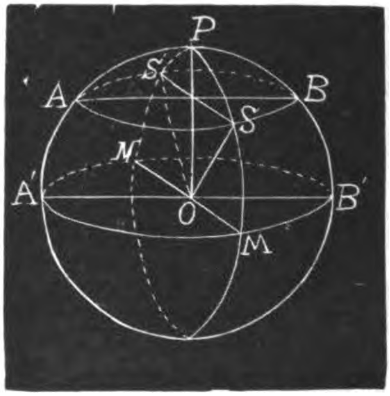
SOLUTIONS OF PROBLEMS.

32. **Proposed by J. F. W. SCHEFFER, Hagerstown, Maryland.**

Suppose it to be possible to perform the passage through the north pole: at what latitude would the maximum distance be saved by a ship sailing on the arc of a great circle instead of a parallel of latitude, the points of departure and destination being 180° apart? Also find the maximum saving.

I. Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland, and E. D. SCALES, Student of Junior Class, University of Mississippi, University, Mississippi.

Put $\angle MOS=\lambda$, $=$the latitude of the *departure-point* of the ship.



According to the problem, the ship departing from S may reach the destination S' by taking either the *longer* route SAS' $=(\pi\cos\lambda)R$, or by taking the *shorter* route $SPS'=(180°-2\lambda)R$. The expression for the number of miles saved by taking the route SPS', becomes $M=2(\lambda-\pi\sin^2\frac{1}{2}\lambda)R$....(1), which is to be a maximum. Differentiating, etc., $\lambda=\sin^{-1}(2/\pi)$, $=39°\ 32'\ 24''.784$; and, consequently, the number of miles saved is $M=[2\sin^{-1}(2/\pi)-\pi+\sqrt{(\pi^2-4)}]R$, $=2620.80359+$ English miles.

II. Solution by O. W. ANTHONY, M. S., Missouri Military Academy, Mexico, Missouri, G. B. M. ZERR, A. M., Staunton, Virginia, and the PROPOSER.

Let λ be the latitude and regard the earth as a perfect sphere with the same volume.

Then the radius of the small circle in latitude λ is $R\cos\lambda$, where $R=$radius of the earth.

$\therefore$ $\pi R\cos\lambda$ is the distance to sail on a parallel of latitude and $(\pi-2\lambda)R$, the distance on a great circle.

$\therefore$ $\pi R\cos\lambda-(\pi-2\lambda)R=$max.

$\therefore$ $\pi\cos\lambda-\pi+2\lambda=$max.

Integrating, we get $\lambda=\sin^{-1}\dfrac{2}{\pi}=39°\ 32'\ 24''.55$. The saving $=R(\pi\cos\lambda-\pi+2\lambda)$. But $R=20902410$ feet according to Col. Clark.

$\therefore$ the saving$=20902410\times.671385=14033564.53785$ feet

$=2657.87207$ miles.

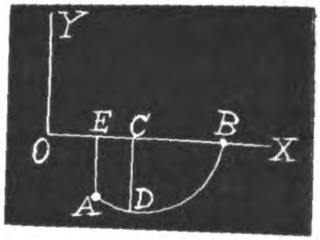
Also solved by *A. H. BELL, Hillsboro, Illinois*, and *E. W. MORRELL, Montpelier Seminary, Montpelier, Vermont.*

33. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, California.

Show that of all curves of a given length, traced on one plane between two given points, and are made to revolve around a common axis situated in that plane, the Catenary generates a minimum area.

I. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let A, B be the two given points, arc $ADB=S$, OBX the common axis, $OB=a$, $OE=b$, $OC=x$, $CD=y$. Then from Calculus of Variations, we get:—



$$\int Vdx=\int_b^a 2\pi y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}dx=\text{a minimum}$$

and $\int V'dx=\int_b^a\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}dx=S.$

$$\therefore DU=D\int(V+\beta V')dx=0.$$

$$V+\beta V'=(2\pi y+\beta)\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}},=\text{also } P_1\frac{dy}{dx}+c$$

$$\text{where } P_1=(2\pi y+\beta)\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-\frac{1}{2}}\frac{dy}{dx}.$$

$$\therefore C+(2\pi y+\beta)\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-\frac{1}{2}}\left(\frac{dy}{dx}\right)^2=(2\pi y+\beta)\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}.$$

$$\therefore dx=\frac{cdy}{\sqrt{(2\pi y+\beta)^2-c^2}}.\quad \text{Let } 2\pi y+\beta=z, \text{ then } dy=\frac{dz}{2\pi}.$$

$$\therefore dx=\frac{c}{2\pi}\frac{dz}{\sqrt{z^2-c^2}},\ x=\frac{c}{2\pi}\log\frac{C}{z-\sqrt{z^2-c^2}}=\frac{c}{2\pi}\log\frac{C(z+\sqrt{z^2-c^2})}{c^2}.$$

$$\therefore x=\frac{c}{2\pi}\log\frac{C\{2\pi y+\beta+\sqrt{(2\pi y+\beta)^2-c^2}\}}{c^2}.$$

$$\therefore y=\frac{1}{4\pi C}\left(c^2e^{\frac{2\pi x}{c}}+C^2e^{-\frac{2\pi x}{c}}\right)-\frac{\beta}{2\pi}, \text{ the equation of a Catenary.}$$

II. **Solution by J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering, A. and M. College, College Station, Texas.**

It is shown in Statics that of all curves of given length between two given points the Catenary has its center of gravity lower than any other—the curve being converse to the axis, and gravity acting at right angles thereto (See Minchin's Statics, Vol. II, page 161).

Now by the theorem of Pappus the area generated equals the length of curve into path described by the center of gravity in turning about the axis; and since the radius of the circle described by the *c. g.* of the Catenary is less than for any other curve considered it is evident the area generated is a minimum.

If the Catenary is concave to the axis of x the area generated will be a maximum, and is so proven in Duhamel's "Elements de Calcul Infinitesimal, Tome Second", page 400.

Prof. MATZ, sent three excellent solutions to the above problem.

PROBLEM.

42. Proposed by J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering, A. and M. College, College Station, Texas.

Show that the volume included between the surface represented by the equation $z=e^{-(x^2+y^2)}$ and the xy plane equals the square of the area of the section made by the zx plane, the limits of x and y being plus and minus infinity.

MECHANICS.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

20. Proposed by CHAS. E. MYERS, Canton, Ohio.

A flexible cord of given length is suspended from two points whose co-ordinates are $(x, y,)$ and (x', y'). What must be the condition of the cord in order that it may hang in the arc of a circle?

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Taking the lowest point for origin, and the horizontal and vertical lines through it for axes of x and y, and $s=a\varphi$......(1) for the intrinsic equation to the circle.

If $\pi=$ the constant horizontal component of tension at all points of the cord, the law of mass as given by Theoretical Mechanics is

$m=\frac{\pi}{g}\frac{d^2y}{dx^2}\Big\backslash\frac{ds}{dx}$....(2). We have $\frac{dy}{dx}=\tan\varphi$, $\frac{dx}{ds}=\cos\varphi$, $\frac{ds}{d\phi}=a$, from (1);

$$\frac{d^2y}{dx^2}=\frac{1}{\cos^2\varphi}\frac{d\varphi}{dx}=\frac{1}{\cos^2\varphi}\frac{d\varphi}{ds}\frac{ds}{dx}=\frac{1}{a\cos^3\varphi}\text{......(3).}$$

Then (2) gives $m=\frac{\pi}{g}\frac{a}{a^2\cos^2\varphi}=\frac{\pi a}{(a-y)^2}$....(4), or the mass unit varies inversely as the square of the distance below the horizontal diameter.

Excellent solutions of this problem were also received from *G. B. M. ZERR, and F. P. MATZ.*

21. Proposed by J. A. CALDERHEAD, Superintendent of Schools, Limaville, Ohio.

Show that, in the wheel and axle, when a force P, acting at the circumference of the wheel, supports a weight Q upon the axle,

$$P.(R\mp\rho\sin\varepsilon)=Q(r\pm\rho\sin\varepsilon)\pm W\rho\sin\varepsilon,$$

where R, r, and ρ are the radii of the wheel, the axle, and their common axis respectively, and ε is the limiting angle of resistance.

Solution By G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia; and F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Let W=weight of wheel and axle, β=the angle between P and Q and also between P and W since Q and W are parallel. The resultant of P, Q, W due to friction is $\pm\sqrt{(Q+W)^2+P^2+2P(Q+W)\cos\beta}\times\rho\sin\varepsilon$.

$\therefore PR=Qr\pm\rho\sin\varepsilon\sqrt{(Q+W)^2+P^2+2P(Q+W)\cos\beta}$.

When $\beta=0$ this becomes $PR=Qr\pm\rho\sin\varepsilon(P+Q+W)$.

$\therefore P(R\mp\rho\sin\varepsilon)=Q(r\pm\rho\sin\varepsilon)+W\rho\sin\varepsilon$.

Also solved by *ALFRED HUME.*

22. **Proposed by DE VOLSON WOOD, C. E., Professor of Mechanical and Electrical Engineering in Stevens Institute of Technology, Hoboken, New Jersey.**

A prismatic bar having a uniform angular velocity ω and a linear velocity of v feet per second, suddenly snaps (by the disappearance of the cohesive force) into an indefinite number of equal parts; required the resultant angular velocity of each piece and the locus of the parts at the end of t seconds after rupture.

Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi.

Take O, the center of gravity of the bar AB, as the origin of a system of rectangular axes, the Y—axis coinciding with the direction of the motion of translation.

Let the motion of rotation be contrary to that of the hands of a clock.

Let the length of the bar be $2nl$, n being the number of equal parts into which it snaps; and let the cross-section and the density, each, be unity.

Denote the middle point of DE, any one of these equal parts, by C, any other point of DE by P.

Let $OC=R$, $OP=r$, and $\angle XOP=\theta$.

At the instant of separation P has a velocity, v, parallel to Y and a velocity, $r\omega$, perpendicular to OB.

The subsequent motion of DE may be determined by supposing it initially at rest and acted upon by such impulsive forces as are expressed in the actual motion at the instant under consideration.

The element of mass at P is acted upon by an impulsive force parallel to Y measured by the momentum $v.dr$, and by a force perpendicular to OB measured by $r\omega.dr$.

Therefore, taking moments about C, the angular velocity of DE, given by the ratio of the moment of the momentum to the moment of inertia, is

$$\frac{\int[v.\cos\theta+r\omega](r-R)dr}{\frac{2}{3}l^3}$$ the limits being $R+l$ and $R-l$.

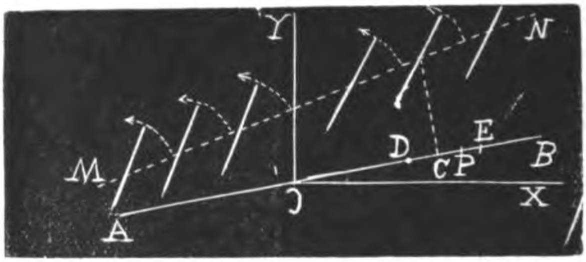
Integrating between these limits, the numerator of this fraction becomes $\frac{2}{3}\omega l^3$.

Hence, after separation, DE will rotate about C with an angular velocity equal to that of the original bar.

C, itself, will move in the direction OY with a velocity $v+R\omega.\cos\theta$ and in the direction XO with a velocity $R\omega.\sin\theta$.

At the end of t seconds the co-ordinates of C' will be given by $x=R\cos\theta-R\omega.\sin\theta.t$ and $y=R\sin\theta+(v+R\ \omega.\cos\theta t)$.

Eliminating R, $y=\dfrac{\tan\theta+\omega t}{1-\omega t.\tan\theta}.x+vt$, or $y=\tan(\theta+\tan^{-1}\omega t)x+vt$.



The locus of the centers of gravity of the parts t seconds after rupture is, therefore, a straight line inclined to X at an angle $\theta+\tan^{-1}\omega t$, and cutting Y at a distance vt from O.

This line coincides with Y after $\dfrac{\cot\theta}{\omega}$ seconds.

When $t=\infty$, the locus is perpendicular to AB.

In the figure the line MN represents the locus, and the arrows the direction of rotation of the parts. The center of gravity of each part moves uniformly in a straight line forever, while the part rotates uniformly about this center of gravity.

This problem was also solved by *F. P. MATZ*.

PROBLEMS.

29. Proposed by J. A CALDERHEAD, A. B., Superintendent of Schools, Limaville, Ohio.

Show that if a body be projected from the angle A of a plane triangle ABC so as to strike the side CB at a point D, then, if its course after reflection at D be parallel to AB, $\tan DAB=\dfrac{(1+\varepsilon)\cot B}{(1-\varepsilon)\cot^2 B}$.

30 Proposed by WILLIAM HOOVER. A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

P is the lowest point on the rough circumference of a circle in a vertical plane at which a partical can rest, friction being equal to the pressure: to find the inclination of the radius through P to the horizon.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

23. Proposed by J. M. COLAW, A. M., Principal of High School, Monterey, Virginia.

Find three positive integral numbers such that the product of the first and the sum of the others is a square and the sum of their cubes is a square.

Solution by J. W. NICHOLSON, A. M., LL. D, President and Professor of Mathematics in the Louisiana State University, and Agricultural and Mechanical College, Baton Rouge, Louisiana.

Let x^2, $2(n^2-1)x^2$, $2(n^2+1)x^2$, be the three integers. The first condition gives $4n^2x^4=\square$; hence we have only to solve

$$\left[x^2\right]^3+\left[2(n^2-1)x^2\right]^3+\left[2(n^2+1)x^2\right]^3=\square,$$

$$\text{or } x^6\left[16n^6+48n^2+1\right]=\square.$$

$\therefore 16n^6=\left(\frac{48}{2}n^2\right)^2$; whence $n^2=36$; and the three integers are x^2, $70x^2$ and $74x^2$, where x is any integer.

Also solved by *H. W. DRAUGHON*, and *G. B. M. ZERR*.

24. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Solve *generally:* The sum of the cubes of n consecutive numbers is a square. Determine the numbers, when $n=2$, $n=3$, $n=4$, and $n=5$.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Let m, $m+1$, $m+2$, $m+3$, etc., represent any consecutive numbers the sum of whose cubes is to be taken.

Solving by the differential method, we obtain

$$S=\frac{n}{4}\left\{n^3+(4m-2)n^2+(6m^2-6m+1)n+4m^3-6m^2+2m\right\}.$$

This reduces to $S=\frac{1}{4}\left\{[n(n+2m-1)]^2+2m(m-1)[n(n+2m-1)]\right\}$, (A).

If the sum of the consecutive cubes is to be a square, then

$$[n(n+2m-1)]^2+2m(m-1)[n(n+2m-1)]=\square=a^2.$$

Adding $[m(m-1)]^2$ to both members, we have

$$[n(n+2m-1)+m(m-1)]^2=a^2+[m(m-1)]^2.$$

This is of the form $(p^2+q^2)^2=(2pq)^2+(p^2-q^2)^2$.

Equating the respective values, and reducing for m and n, we obtain $2m=1+\sqrt{4p^2-4q^2+1}$, and $2n=\sqrt{4p^2+4q^2+1}-\sqrt{4p^2-4q^2+1}$. It will be observed that each of the radical quantities is of the form of an odd square, $4p+1$.

There are two conditions that will render the radicals rational, and, at the same time, have a an integer:—

(1) When $4q^2=4p$. Then $m=p=q^2$, and $n=1$. According to this condition there is but *one cube* that can be taken at one time, and hence there would be no *sum of cubes*. This cube is the cube of a square, and is, therefore, also the square of a cube.

(2) The second condition is when $p^2=q^2$. Then $m=1$; and substituting this value in (A), we obtain $S=\left\{\frac{n(n+1)}{2}\right\}^2$, which is the square of the sum of the series, $1+2+3+\ldots\ldots+n$. From this, then, we have $1^3+2^3+3^3+\ldots.+n^3=(1+2+3+\ldots.+n)^2$, or *the square of the sum of the first n natural numbers is equal to the sum of their respective cubes.*

Therefore, in order that the sum of the cubes of n consecutive numbers be a square, *the first number must be unity.*

When $n=2$, the numbers are 1 and 2; when $n=3$, the numbers are 1, 2, and 3; &c., &c.

Also solved by *Professor COOPER D. SCHMITT, and the PROPOSER.*

II. **Solution by B. F. FINKEL, A. M., Professor of Mathematics in Kidder Institute, Kidder, Missouri.**

Let $S=1+2+3+\ldots\ldots+n=(n+1)n/2$;
$S_2=1^2+2^2+3^2+\ldots\ldots+n^2=n(n+1)(n+2)/6$; and
$S_3=1^3+2^3+3^3+\ldots\ldots+n^3=?$

Now

$$\begin{aligned}
(n+1)^4-n^4 &= 4n^3+6n^2+4n+1\\
n^4-(n-1)^4 &= 4n^3-6n^2+4n-1=4(n-1)^3+6(n-1)^2+4(n-1)+1\\
(n-1)^4-(n-2)^4 &= 4n^3-18n^2+28n-15=4(n-2)^3+6(n-2)^2+4(n-2)+1\\
(n-2)-(n-3)^4 &= 4n^3-30n^2+76n-65=4(n-3)^3+6(n-3)^2+4(n-3)+1\\
\ldots\ldots &= \ldots\ldots=\ldots\ldots\ldots\ldots\\
5^4-4^4 &= 369 \quad =4\times4^3+6\times4^2+4\times4+1\\
4^4-3^4 &= 175 \quad =4\times3^3+6\times3^2+4\times3+1\\
3^4-2^4 &= 65 \quad =4\times2^3+6\times2^2+4\times2+1\\
2^4-1^4 &= 15 \quad =4\times1^3+6\times1^2+4\times1+1\\
1^4-0^4 &= 1 \quad =4\times0^3+6\times0^2+4\times0+1
\end{aligned}$$

Adding, $(n+1)^4=4S_3+6S_2+4S+n+1$.

Whence,

$$\begin{aligned}
S_3 &= [(n+1)^4-n-1-6S_2-4S]/4,\\
&= [(n+1)^4-n-n(n+1)(n+2)-2(n+1)n]/4,\\
&= [n^4+2n^3+n^2]/4=[\tfrac{1}{2}n(n+1)]^2.
\end{aligned}$$

If $n=2$, $S_3=9$; if $n=3$, $S_3=36$; if $n=4$, $S_3=100$; if $n=5$, $S_3=225$.

NOTE.—The above method is useful in summing the series: $1^r+2^r+3^r+4^r\ldots\ldots n^r$, where r is any integer.

PROBLEMS.

32. Proposed by A. H. BELL, Hillsboro, Illinois.

Decompose into its prime factors the number 549755813889.

33. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find three different *sixth powers* whose sum is a square.

[The solution of this problem, if possible, is an answer to the *note* under the solution of Prob. 16.]

AVERAGE AND PROBABILITY.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

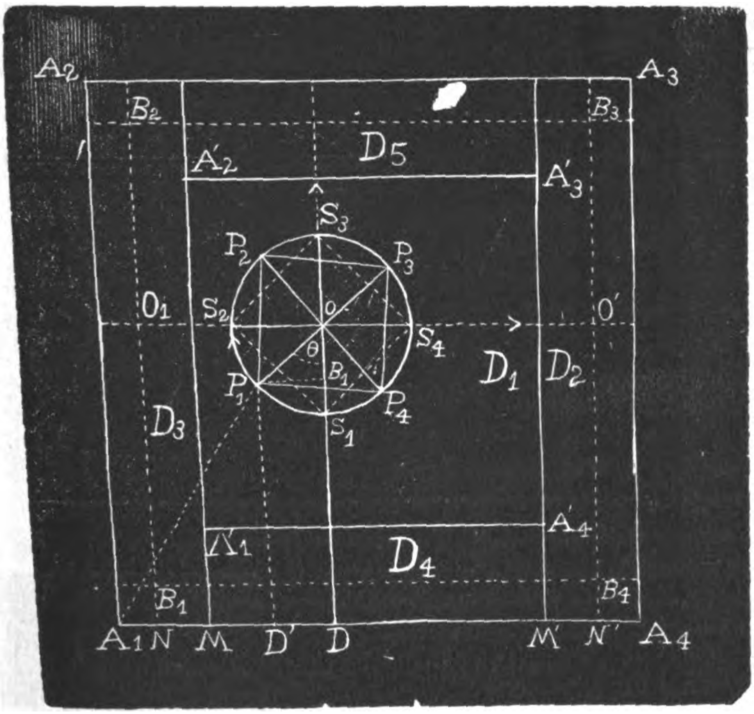
SOLUTIONS OF PROBLEMS.

20. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

A surface one inch square is thrown at random upon a surface one foot square, but in such a manner as always to lie wholly upon the larger surface. Find the mean value of *the sum of the distances* of the vertices of the smaller surface, from *any* vertex of the larger surface.

Solution by the PROPOSER.

Let $S_1S_2=s$, $A_1A_2=12s$, $AD=x$, $DO=y$; then $S_1O=\frac{1}{2}s\sqrt{2}$. Put $\angle S_1OP_1=\angle S_2OP_2=\angle S_3OP_3=\angle S_4OP_4=\theta$; then $DD'=\frac{1}{2}s\sqrt{2}\sin\theta$, and $(DO-P_1D')=\frac{1}{2}s\sqrt{2}\cos\theta$.



$\therefore$ $A_1D'=\frac{1}{2}(2x-s\sqrt{2}\sin\theta)$, and $P_1D'=\frac{1}{2}(2y-s\sqrt{2}\cos\theta)$; also, $\triangle_1=A_1P_1=\frac{1}{2}\sqrt{[(2x-s\sqrt{2}\sin\theta)^2+(2y-s\sqrt{2}\cos\theta)^2]}$, $\triangle_2=A_1P_2$ $=\frac{1}{2}\sqrt{[(2x-s\sqrt{2}\cos\theta)^2+(2y+s\sqrt{2}\sin\theta)^2]}$, $\triangle_3=A_1P_3=\frac{1}{2}\sqrt{[(2x+s\sqrt{2}\sin\theta)^2+(2y+s\sqrt{2}\cos\theta)^2]}$, $\triangle_4=A_1P_4=\frac{1}{2}\sqrt{[(2x+s\sqrt{2}\cos\theta)^2+(2y-s\sqrt{2}\sin\theta)^2]}$.

Let $\triangle=(\triangle_1+\triangle_2+\triangle_3+\triangle_4)$; then five cases confront us, for consideration.

First Case—The point O may lie in the square surface $A'_1A'_2A'_3A'_4$. If $\frac{1}{2}(24-\sqrt{2})s=a$ and $\frac{1}{8}s\sqrt{2}=b$, the mean value of the sum of the distances in this case becomes

$$\mathbf{D}_1=\int_b^a\int_b^a\int_0^{\frac{1}{4}\pi}[\triangle_1+\triangle_2+\triangle_3+\triangle_4]dxdyd\theta\div\int_b^a\int_b^a\int_0^{\frac{1}{4}\pi}dxdyd\theta\ldots.(1).$$

Second Case—The point O may lie in the rectangular surface D_2; then if $11\frac{1}{8}s=c$, $\frac{1}{8}s=e$, and $\sin^{-1}(x/e\sqrt{2})=\phi$, the mean value in this case becomes

$$\mathbf{D}_2=\int_a^c\int_e^c\int_0^\phi[\triangle_1+\triangle_2+\triangle_3+\triangle_4]dxdyd\theta\div\int_a^c\int_e^c\int_0^\phi dxdyd\theta\ldots.(2).$$

Third Case—The point O may lie in the rectangular surface D_3; and in this case,

$$\mathbf{D}_3=\int_e^b\int_e^c\int_0^\phi[\triangle_1+\triangle_2+\triangle_3+\triangle_4]dxdyd\theta\div\int_e^b\int_e^c\int_0^\phi dxdyd\theta\ldots.(3).$$

Fourth Case—The point O may lie in the rectangular surface D_4. Put $\frac{1}{4}s(23-\sqrt{2})=f$, and $\frac{1}{8}s(\sqrt{2}+1)=g$; then the mean value in this case becomes

$$\mathbf{D}_4=\int_g^f\int_e^b\int_0^\phi[\triangle_1+\triangle_2+\triangle_3+\triangle_4]dxdyd\theta\div\int_g^f\int_e^b\int_0^\phi dxdyd\theta\ldots.(4).$$

Fifth Case—The point O may lie in the rectangular surface D_5; then the mean value in this case becomes

$$\mathbf{D}_5=\int_g^f\int_a^c\int_0^\phi[\triangle_1+\triangle_2+\triangle_3+\triangle_4]dxdyd\theta\div\int_g^f\int_a^c\int_0^\phi dxdyd\theta\ldots.(5).$$

Consequently the *required* mean value becomes

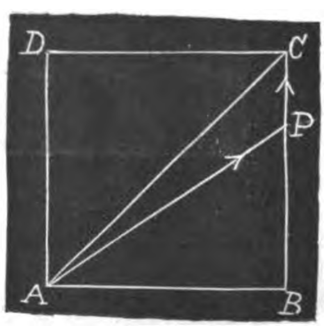
$$\mathbf{D}=\tfrac{1}{5}(\mathbf{D}_1+\mathbf{D}_2+\mathbf{D}_3+\mathbf{D}_4+\mathbf{D}_5)\ldots.(6);$$

and the labor required in the performance of the integrations indicated is simply enormous.

21. Proposed by H. W. DRAUGHON, Clinton, Louisiana.

From one corner of a square field, a boy runs in a random direction, with a random uniform velocity. The greatest distance the boy can run in one minute is equal to the diagonal of the field. What is the probability that the boy will be in the field at the end of one minute?

Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.



Let $AB=a$, and $BP=x$; then $AP=\sqrt{(x^2+a^2)}=w$, and $AC=a\sqrt{2}=m$. The boy will be in the field at the expiration of $t=1$ minute, if v be not greater than $\sqrt{(x^2+a^2)}$. Hence the required probability becomes

$$\mathbf{P}=\int_0^a\int_0^w dxdv\div\int_0^a\int_0^m dxdv,=\frac{1}{a^2\sqrt{2}}\int_0^a\sqrt{(x^2+a^2)}dx$$

$$=\frac{1}{a^2\sqrt{2}}\left[\frac{x\sqrt{(x^2+a^2)}}{2}+\frac{a^2\log[x+\sqrt{(x^2+a^2)}]}{2}\right]_o^a=\tfrac{1}{2}\left[1+\frac{\log(\sqrt{2}+1)}{\sqrt{2}}\right].$$

O. W. Anthony gets as a result $\frac{2}{\pi}$. Professor *MATZ* furnished two solutions.

22. Proposed by ALTON L. SMITH, Instructor in Drawing, Polytechnic Institute, Worcester, Massachusetts.

In a series of counts of the votes on a legislative act relative to the city of Worcester, the following results were obtained:

	YES	NO
1st count	5566	5511
2nd "	5519	5558
3rd "	5546	5517
4th "	5512	5551
5th "	5512	5541

What is the probability that the last count (the 5th) is correct?

Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor, College, New Windsor, Maryland.

I. Since the counts taken independently must be either correct or incorrect the probability that the fifth count is *incorrect* is

$P'_5=\frac{1}{2}/(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2})=\frac{1}{5}$. Hence the probability that the fifth count is *correct* is $P_5=1-P_5'=\frac{4}{5}$.

II. According to *The Law of Experience*, the second count should show a *greater* probability as to correctness and a *smaller* probability as to incorrectness than the first count shows; that is, the probability as to the correctness of the fifth count should be greater than is the *similar* probability with respect to any other count *lower* than the fifth. This Law, according to the notation adopted, gives $P_1'=\frac{1}{2}, P_2'=\frac{1}{4}, \ldots, P_5'=\frac{1}{32}=(\frac{1}{2})^5$. Hence $P_5=1-(\frac{1}{2})^5=\frac{31}{32}$, which is the probability that the fifth count is correct.

PROBLEMS.

29. Proposed by JOHN DOLMAN, Jr., Philadelphia, Pennsylvania.

Neglecting perturbations, what is the average distance of the earth from the sun?

30. Proposed by F. P. MATZ, M. A., M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Find the average area of all the triangles that can be inscribed in a given circle.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

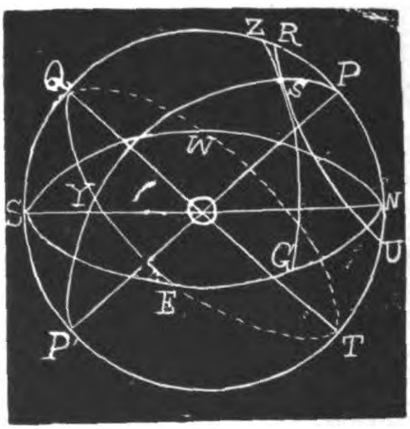
SOLUTIONS OF PROBLEMS.

17. Proposed by SAMUEL HART WRIGHT, M. D., M. A., Ph. D., Penn Yan, Yates county, New York.

A bright star passed my meridian at 7 P. M. The Chronometer soon after ran down and stopped, but I set it again when the same star had a true altitude of $30°=\alpha$. What time was it then, my latitude being $42° 30' N.=\lambda$, and the star's Declination $60° N.=\delta$?

Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let O be the place of the observer, Z his zenith, $SENW$ the horizon, P and P' two poles of the heavens, $EQWT$ the celestial equator. S the star. YS the star's declination. $\angle SPZ$ the star's hour angle=arc QY, $SZPN$ the meridian. ZSG arc of star's vertical arc, SG star's altitude. From the spherical triangle ZPS we get $\cos ZS=\cos PS \cos PZ+\sin PS \sin PZ \cos ZPS$. But $ZS=90°-\alpha$, $PS=90°-\delta$, $PZ=90°-\lambda$, $\angle ZPS=h$.



$$\therefore \cos h=\frac{\sin\alpha-\sin\lambda\sin\delta}{\cos\lambda\cos\delta}$$

$$=\frac{\sin 30°-\sin 42\tfrac{1}{2}°\sin 60°}{\cos 42\tfrac{1}{2}°\cos 60°}$$

or we might use the formula

$$\sin^2\tfrac{1}{2}h=\tfrac{1}{2}\,\frac{\cos(\lambda-\delta)-\sin\alpha}{\cos\lambda\cos\delta}$$

$$=\tfrac{1}{2}\frac{\cos 17\tfrac{1}{2}°-\sin 30°}{\cos 42\tfrac{1}{2}°\cos 60°}.\quad h=103°\ 20'\ 37''.93$$

$=6$ hr. 53 m. 22.53 sec.

$\therefore$ time$=53$ m. 22.53 sec. after 1 o'clock A. M.

In the figure the star is east of the meridian while in its true position it is west of the meridian.

Also solved by *Professors MATZ, SCHEFFER, WHITAKER and the PROPOSER.*

18. Proposed by M. C. STEVENS, A. M., Professor of Mathematics, Purdue University, Lafayette, Indiana.

"Show generally that a system of confocal conics is self-orthogonal."—*Johnson's Differential Equations.*

Solution by WILLIAM WOOLSEY JOHNSON, M. A., Member of the London Mathematical Society, and of the American Mathematical Society, Professor of Mathematics in the United States Naval Academy, Annapolis, Maryland.

The equation of a system of confocal conics, foci at $(\pm c, 0)$, is $\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1\ldots.(1)$, a being the arbitrary constant. Differentiating, and putting p for dy/dx, $\frac{x}{a^2}+\frac{yp}{a^2-c^2}=0$, or $a^2(x+yp)=c^2x\ldots.(2)$. Eliminating a between (1) and (2), $\frac{x(x+yp)}{c^2}-\frac{y(x+yp)}{c^2p}=1$, or $xyp^2+(x^2-y^2-c^2)p-xy=0$ ....(3), which is the differential equation of the system. Putting $p=-\frac{1}{p'}$, we have $xyp'^2+(x^2-y^2-c^2)p'-xy=0\ldots.(4)$, which is the equation of the trajectory when $p'=dy/dx$, and it is identical with equation (3). In other words, the roots of equation (3), as a quadratic for p, are negative reciprocals.

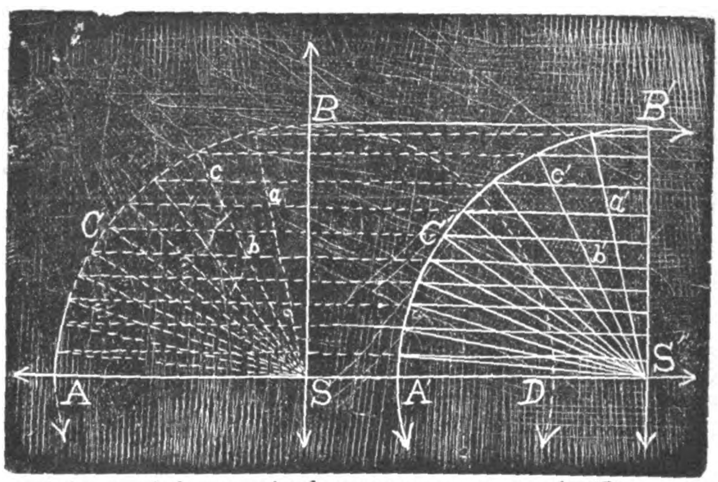
Also solved by *Professors MATZ and ZERR.*

19. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

A spherical shrapnel-shell is moving in a horizontal direction, with a constant velocity $V_1=1500$ feet per second. The shell explodes at a height $h=\frac{1}{2}g$; and the fragments of the shell and the balls inclosed by the shell, are *equably* scattered with a uniform velocity $V_2=1200$ feet per second. Draw the curve bounding *the minimum surface on the earth* on which the fragments of the shell and the balls inclosed by the shell, have fallen.

Solution by the PROPOSER.

The path of the shell is an inverted catenary of equal (or uniform) strength. At the point of maximum altitude of the path, $h=\frac{1}{2}g$, the shell moves in a horizontal direction. Ignore all such *complicating factors* as the resistance of the atmosphere, the rotary motion and geodesic contour of the earth, etc., as practically nugatory in



so far as the position and form of the required curve are concerned. The range of the shell can now be determinately or approximately deduced. The determination of the *maximum range* of the fragments of the shell and the balls inclosed by the shell, is the next step preliminary to the drawing of the required curve. In the problem under consideration, the position of the shell at the instant of explosion is at some (unrepresented) point P vertically above S and perpendicular to the plane of the circle whose center is at S; that is $PS=h=\frac{1}{2}g$ $=16\frac{1}{12}$ feet. The fragments a, b, c, resulting from the explosion of the shell, fall on the earth at a', b', c'. The required curve, therefore, is a circle, as it should be; and the radius of this circle is the maximum range of the *fragments* and *balls*. Of course, if the atmospheric resistance and other complicating factors be not ignored, it is reasonable that the required curve should be somewhat *oval-shaped*—thus, to a certain extent, resembling the apparent disc of the rising sun or of the rising full moon.

20. **Proposed by SAMUEL HART WRIGHT, M. D., M. A., Ph. D., Penn Yan, Yates County, New York.**

When does the Dog-Star and the Sun *rise together* in latitude $\lambda=+42^\circ\ 30'$, if the Right Ascension of the said star be $\alpha=6$ hrs., 40 min., 30 sec., and the declination $\delta=-16^\circ\ 33'\ 56''$?

Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

In order that "The Dog Star" and the Sun may rise together, their *hour-angles at the instant of rising* must be the same. According to *Chauvenet's Spherical and Practical Astronomy*, Vol. I., p. 218, Art. 153, we have the formula:

$$t=\pm[-\tan\lambda\tan\delta],=74^\circ\ 10'\ 57''.77,=4h.\ 56m.\ 43.851\ sec.$$

Therefore, the next two *critical dates;* that is, the next two dates on which the *cosmical rising* of "The Dog Star" is possible in latitude $\lambda=+42^\circ\ 30'$, must be May 2, 1895, and August 2, 1895. By the Right Ascension of "The Dog-Star," as given in the problem, we are led to consider August 2, 1895, as the required date.

NOTE—The next two critical dates with respect to the *cosmical setting* of "The Dog-Star," evidently, are May 9, 1895, and August 11, 1895; and of these dates, the required one is May 9, 1895.

Also solved by *Professor ZERR* and the *PROPOSER*.

NOTE—No one of our contributors has as yet been able to effect a full and satisfactory solution to problem 21.

OUTLINE OF INVESTIGATION FOR ASYMPTOTES.

(Continued from page 183.)

In $Y=y-x\dfrac{dy}{dx}$, sub. value of $\dfrac{dy}{dx}$, gives

$$Y=\frac{4ay^2-5xy^2-3x^3}{4ay-2xy}=\frac{4a-5x-\dfrac{3x^3}{y}}{\dfrac{4a}{y}-\dfrac{2x}{y}}=\frac{4a-5x}{0}=\frac{q}{0}=\infty,$$

since $y\doteq\infty$ at limit.

Then by(c) under B, sub. values of X and Y, in $\dfrac{x}{a}+\dfrac{y}{b}=1$, gives

$\dfrac{x}{2a}+\dfrac{y}{\infty}=1.\ \therefore\ \dfrac{x}{2a}+0=1.\ \therefore\ x=2a$ which is the equation sought.

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

Papers for the Mathematical Congress at Kazan.

On the occasion of the dedication of the Lobachevski monument at Kazan will be held a mathematical congress of a week's duration.

It is very much desired by the managment that some papers may be contributed by Americans.

As a complete program of the scientific communications to be made in the sessions will be issued this coming February, it is not too early to solicit American scientists to think of preparing something for this memorable ocoasion.

Dr. George Bruce Halsted has been asked by President Vasiliev to act for him in this matter, to correspond on questions of detail with any who hope to attend the Congress in person, to take charge of the communications of those who do not anticipate being present and to guarantee their proper presentation.

Note on the May Number of the American Mathematical Monthly.

By GEORGE BRUCE HALSTED.

Mr. Warren Holden cites Halsted's Lobachevski's Geometry, but evidently has not read it, since he makes the common *petitio principii* of assuming that the equidistantial is a straight line. Mr. J. N. Lyle is as usual hopelessly muddled. Every one else knows that Beltrami proved Lobachevski's triangle and two-dimensional geometry can exist not only in pseudo-spherical space but also in Euclidean space. To apply to the immortal Helmholtz the sentence "This performance is plainly pseudo-logical," is simply sickening.

EDITORIALS.

Between July 1st and August 10th address all communications to B. F. Finkel, Chicago University.

We are pained to note the death of Dr. Daniel Kirkwood which took place at Los Angles, California, June 11. For a brief sketch of Dr. Kirkwood see the May No., Vol. I.

Editor Finkel has been elected Professor of Mathematics and

Physics in Drury College, one of the best institutions of learning in the State of Missouri. During the summer, Professor Finkel will attend the Chicago University at which institution he has been assigned a graduate Scholarship.

ON JUNE 6, from New Windsor College, Professor Hudson A. Wood, of Stevens Institute of Technology, New Jersey, because of great mathematical knowledge, success in mathematical teaching, invention of a *Perpetual Calendar* (published by A. S. Barnes & Co., New York) and manuscript of an extensive treatise on Plane and Spherical Trigonometry—book soon to be published; and Professor G. B. M. Zerr because of his high mathematical attainments as evinced by numerous contributions to the MONTHLY and on account of a thesis on "The Centroid of Surfaces," received *Cum summa laude* the degree of *Doctor of Philosophy*.

WE ARE pleased to note that our valued contributor, Professor G. B. M. Zerr, has been elected Vice-President of the Inter State College, Texarkana, Texas. He will also fill the chair of Mathematics and Sciences in that institution. Professor Zerr has lately received the degree of Ph. D. from a reputable College.

PROFESSOR W. W. LANDIS, of Thiel College, Greenville, Pennsylvania, has just been elected Professor of Mathematics in Dickinson College, Carlisle, Pennsylvania. Professor Landis always takes a lively interest in the MONTHLY.

THERE will be no MONTHLY issued during the month of July. A double number, July-August No., however, will be issued in August.

BOOKS AND PERIODICALS.

Elements of Plane and Solid Geometry: By John Macnie, A. M., Author of "Theory of Equations." Edited by Emerson E. White, A. M., LL. D., Author of *White's Series of Mathematics*. 8vo. cloth and leather back. 374 pp. Price, $1.25. Chicago and New York: American Book Company.

This text-book on geometry is worthy a place among the very best works published in America. We are sorry to note that this late work contains an erroneous demonstration of the proposition: *Triangular pyramids having equivalent bases and equal altitudes are equivalent.* This erroneous demonstration should not consign the book to oblivion as there are but very few texts on geometry that have a correct demonstration. For a discussion of the fallacy in the demonstration see p. 67 March No., Vol. I. A very commendable feature of this book is the large collection of well selected original propositions, there being 972 scattered throughout the book. We recommend it to all teachers needing a good book on geometry.

A Practical and Complete English Grammar. By J. G. Park, Instructor in English Grammar, Logic, Mental and Moral Philosophy, Ohio Normal University, Ada, Ohio. 8vo cloth. 274 pp. Price, $0.65. New York and Chicago: American Book Co.

The first 49 pages of this grammar are devoted to Language Lessons. In the remaining part of the book, Professor Park has given to the general teaching public, his admirable method of teaching grammar. Those who are personally acquainted with his method know that, for power of creating interest and enthusiasm in classes ranging from 300 to 350 students, it is unsurpassed. We believe that this grammar will be epoch-making in the teaching of grammar. As to the real technique of grammatical constructions, there may be some points on which the best philologists will differ from Prof. Park, but upon the whole his book will bear careful perusing. We trust that this book may supplant the many useless and fossiliferous texts that are preventing a thorough mastery of the English Language and creating a disgust for its study in many of the schools in this country.

The Review of Reviews. An International Monthly Magazine. Edited by Albert Shaw. Price, $2.50 per year. Single Number, 25 cents. The Review of Reviews Co.: New York City.

This Magazine is one of the best of our day. All the current events of the world are briefly noted while the more important events are thoroughly discussed. the leading articles in the leading magazines of the world are briefly reviewed. The June No. of the Review of Reviews contains nearly the whole of Dr. Halsted's biography of Prof. Cayley. This extract of the biography was taken from the April No. of the MONTHLY.

American Journal of Mathematics. Edited by THOMAS CRAIG, with the co-operation of SIMON NEWCOMB. Quarterly. 4to. $5 per volume. Johns Hopkins Press. Baltimore, Maryland.

The July number of this valuable publication contains the following papers: "On Irrational Covariants of certain Binary Forms," by E. Study; "On the Connection between Binary Quartics and Elliptic Functions," by E. Study; "Semi-Combinants as Concomitants of Affiliants," by Henry S. White; "Simplification of Gauss's third Proof that every Algebraic Equation has a Root," by Maxime Bocher; "Note sur lignes cycloidales," par Rene de Saussure; "Note on Lines of Curvature," by Thomas Hardy Taliaferro.

Annals of Mathematics. Ormond Stone, Editor, University of Virginia. The January number contained "Literal Expression for the Motion of the Moon's Perigee," by G. W. Hill; "Note on Gregory's Discussion of the Treatment of the Conditions for an Umbilicus," by Angelo Hall; "Concerning the Definition by a system of Functional Properties of the Function $f(z)=\frac{\sin \pi z}{\pi}$, by E. H. Moore. Twelve problems are solved, and seventeen exercises proposed.

Journal de Mathematiques Elementaires. Public par H. Vuibert. Paris: Librairie Nony et Cie 17, rue des Ecoles.

The May number contains the usual amount of interesting matter, including the solutions of problems in Arithmetic, Algebra, Geometry, Trigonometry, and Physics. Several new problems are proposed.

L'Intermediaire des Mathematiciens. Dirige par C.-A. Laisant, et Emile Lemoine. Monthly. Paris.

The May number of this interesting Journal contains Questions 550 to 570, and publishes ten pages of answers to questions previously proposed.

El Progreso Matematico. Periodico de Matematicas Puras y Aplicadas. Director: Don Zoel G. de Galdeano, catedratico de la Universidad de Zaragoza.

The March number of this excellent Monthly has several interesting papers, and many fine problems and solutions.

Periodico di Matematica per L'Insegnamento Secondario. Publicato per cura di Aurelio Lugli, Professore di Matematica nel R. Instituto tecnico di Roma. Via Panisperna, 69-Roma.

In the January-February number several important papers are published, and quite a number of questions are proposed and many others solved.

The Educational Times, London. for June has the usual amount of space devoted to its Mathematical Department. Fifteen solutions appear, and 37 new problems are proposed, several of which are republished from the MONTHLY.

The Kansas University Quarterly for April has been received. Nine very interesting articles are published, but no mathematical contributions appear in this number.

The Monist. The April number contains several important papers by well-known writers. Nearly every issue of this valuable Quarterly has one or more contributions of a mathematical character.

Miscellaneous Notes and Queries. A Monthly Magazine of History, Folk-Lore, Mathematics, Mysticism, Art, Science, etc.

The June Issue contains much varied and interesting information gathered from many sources.

NOTE:—In the cut used in the fifth solution on page 190, by mistake the letter P was omitted at a point on the upper part of the line AD. Readers of the MONTHLY will please note this error.

JOHN HENRY LAMBERT.

THE

AMERICAN

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VOL. II. JULY-AUGUST, 1895. NO. 7-8.

BIOGRAPHY.

JOHN HENRY LAMBERT.

BY DR. GEORGE BRUCE HALSTED.

JOHN HENRY LAMBERT was born August 29, 1728, at Muehlhausen in Alsace. Through unaided study he mastered the then ordinary mathematics. In 1745 he became secretary to Professor Iselin editor of a newspaper at Basel, who three years later obtained his appointment as private tutor to the family of the Swiss President A. von Salis of Coire.

After completing with his pupils a tour of two years duration through Goettingen, Utrecht, Paris, Marseilles, and Turin, he resigned his position in 1759 and settled at Augsburg. Afterward Munich, Erlangen, Coire, Leipzig became for brief successive intervals his home.

Finally in 1764 he settled in Berlin, and was elected a member of the Royal Academy of Sciences.

He died of consumption on September 25th, 1777. Among his most remarkable works are *Pyrometrie* (Berlin, 1779) a systematic treatise on heat; *Photometria* (Augsburg, 1760); *Insigniores orbitae Cometarum proprietates* (Augsburg, 1761). Beitrage zum Gebrauche der Mathematis und deren Anwendung (4 Vols. Berlin, 1765-72).

It is to Lambert, another Leibnitz as to the variety and depth of his attainments and views, that is due the law of the conduction of heat in a small metallic bar exposed at one end to the constant action of a source of heat.

Experiment confirmed his theoretic results, which become the basis of further researches and of the tremendously important Theory of heat of the mathematician Fourier.

Though belonging to a time when the wonder-working Calculus lus and analytic mathematics dazzled all, yet Lambert, though the associate at

Berlin of Lagrange and Euler, nevertheless retained the taste for pure geometry and understood how to make of it the most surprising applications. He demonstrated synthetically his celebrated theorem on the arcs of ellipses of the same major axis, described in the same time. His *Freye Perspective*, published in 1759, and increased by a second part in 1773, makes use of this theory as of a geometric method and proves many theorems relating to the descriptive properties of figures. He is the true forerunner of Monge. Again, the problem of the arithmetical quadrature of the circle is as old as mathematics.

Lambert it was who first proved the task of the π-computers endless by demonstrating that π is irrational. This alone would have made him immortal. The proof communicated by Lambert to the Berlin Academy of Sciences in 1761 is reproduced in Note IV. of Legendre's *Geometrie*.

Lambert developed De Moivre's theorems on the trigonometry of complex variables and introduced the hyperbolic functions which he designated by $sinh\,x$, $cosh\,x$, &c.

His work must therefore have attracted the especial attention of Gauss, one of whose chief claims is the development of the theory of complex variables.

In still another direction Lambert deserved immortality. He was the originator of Symbolic Logic.

[See my article *Symbolic Logic* in Johnston's New Universal Encyclopaedia]. He fully recognized that the four algebraic operations, addition, subtraction, multiplication, division, have each an analogue in logic, namely *Zusammensetzung*, *Absonderung*, *Bestimmung*, *Abstraction*, which may be symbolized by +, −, ×, ÷. He also preceived the *inverse* nature of the second and fourth as compared with the first and third. He enunciates with perfect clearness the principal logic laws, such as the commutative and distributive. He *develops* simple logical expressions precisely as Boole did later. He interpreted and represented hypothetical propositions precisely as Boole did. In one passage at least he recognized that the inverse process, marked by division, is an *indeterminate* one.

Venn says: "To my thinking he and Boole stand quite supreme in this subject in the way of originality." Even Kant calls Lambert "*der unvergleichliche Mann.*"

In the very short and imperfect sketch of Lambert by F. W. Cornish of Eton College inserted in the Encyclopaedia Britannica in 1882 we read: "In Bernouilli and Hindenburg's *Magazin* (1787-1788) he treats of the roots of equations and *of parallel lines.*" From this lover of pure geometry, this prophetic pioneer in logic, in mathematic, in their combination, that could only mean the non-Euclidean geometry. And so it was.

The essay, "Zur Theorie der Parallellinien," was written in September, 1766, but first published in 1786 by F. Bernouilli (a kinsman of John Bernouilli) from the papers left by Lambert, and appears in the *Leipziger Magazin fuer reine und angewandte Mathematik*, herausgegeben von J. Bernoulli und C. F. Hindenburg, erster Jahrgang, 1786, Seite 137 ff.

In this remarkable work Lambert maintains:

(1) The Parallel-Axiom needs a proof, since it does not hold for the geometry on a sphere.

(2) In order to bring before the perceptive intuition a geometry in which the triangle's angle-sum is less than the two right angles we need the help of an "imaginary sphere."

(3) In a space in which the triangle's angle-sum is different from two right angles, there is an absolute measure [a natural unit for length].

Just as Lambert's celebrated memoir, "Vorlaeufige Kentnisse fuer die, so die Quadratur und Rectification des Circuls suchen," closed a question whose history comprises four thousand years, and which therefore pertains to the very oldest problems of mankind, so this essay of Lambert "Zur Theorie der Parallellinien" should have ended the equally fruitless and almost equally ancient striving after a proof of the Postulatum of Euclid, the ordinary assumption for the treatment of parallels, or anything equivalent to it. Will one man ever again wound to the death two such dragons feeding for centuries on human brains!

In a review of my translation of Vasiliev's Address on Lobachevski Mr. Charles S. Pierce says in *The Nation*, April 4th, 1895; "However, Gauss was not the first discoverer. Lambert in 1785, in a printed book, spoke plainly of a space where the angles of a triangle should sum up to less than 180 degrees, and mentions one of its most remarkable properties. Guass most likely knew of this."

Should this be so, the last claim left to be made for Guass in the determined and persistent endeavor of his German admirers to keep him prominently figuring in the history of the greatest achievment of modern culture, the non-Euclidean geometry, namely the claim that he was the first to recognize with complete clearness the uselessness of all attempts to prove the eleventh axiom, becomes meaningless.

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER. Ph. D., Professor of Mathematics, University of Michigan, Ann Arbor, Michigan.

(*Continued from the June Number.*)

In what precedes we have endeavored to convey some of the most important general concepts in regard to the theory of substitution groups, we proceed to some more special concepts leading up to the branch of this subject which we desire to pursue first, viz. the *construction* of the substitution groups containing a given number of letters.

We shall not aim at a rigorous development of any part of the subject but rather at conveying some of the most fruitful concepts and thus to aid (1) those who desire to form a good general idea of the subject without entering into details and (2) those who desire to acquaint themselves with such facts and methods as may be helpful in a more thorough study. The methods used will in general, be such as lead to a rigorous development, but, for the sake of simplicity we shall not pursue them sufficiently far to call our treatment a rigorous development. "In presenting any mathematical theory it is in general, not so important that a rigerous presentation of the subject is given throughout as that the methods used are those which serve as a means for a rigorous presentation.*

The theory of groups is divided into two parts, viz. *the theory of substitution groups* and *the theory of transformation groups*. To these we might perhaps add *the theory of abstract groups*, which was first suggested by Cayley. Its foundation was laid by Dyck in his noted articles in the Mathematische Annalen for 1882 and 1883. This theory has not yet been extensively developed. The theory of transformation groups was founded by Sophrus Lie who began to publish his results in 1870. It bears the same relations to the solution of differential equations as the theory of substitution groups bears to the solution of the algebraic equations. It has not yet had time to take a definite station among the other branches of mathematics but the indications are that it will occupy a very high one.

The theory of substitution groups is considerably older than the two just named. It might be said to have been founded by Abel† but traces of it are found much earlier, as for instance in the writings of Lagrange.§ The most prominent among those who have contributed to its development are Cauchy, Galois and Jordan. Jordan's "Traite des Substitutions et des Equations" is the standard treatise. After this come Netto's "Substitutionentheorie und ihre Anwendungen auf die Algebra" and the chapters on this subject in Serret's "Cours d'Algebre Superieure," second volume. The only work on the subject in the English language is Professor Cole's translation of Netto's work. A great part of the knowledge of this, as well as of most other modern mathematical subjects, is to be found only in the journals.

Having found the place which the theory of substitution groups occupies in the general theory of groups we proceed to locate the theory of the construction of groups in the more general theory of substitution groups. Since groups are merely instruments of operation it follows that the study of the construction of groups must be the study of instruments and not the more useful study of their uses and the methods of using them. It is evident that the study of the use and the modes of using an instrument can, as a rule, be prfitably pursued only after a thorough acquaintance had been formed with the instrument itself and it therefore seems proper for us first to pay attention to its study.

Another advantage in beginning in this way is the fact that a thorough

* Weberhaupt dürfte es ja bei der Darlegung einer mathematischen Theorie wenger auf eine durchweg strenge Darstellung, als villmehr darauf ankommen, dass die angegebenen methoden die zur strengen Darstellung erforderlichen mittal gewahren. Neumann, S. VIII Riemann's Theorie der Abel'schen Integrale.

†Crelle's Journal für Mathematik, vol. I. 1826

§Dr. James Pierpont: Bulletin of the American Mathematical Society, May, 1895.

acquaintance with a complex instrument is apt to suggest uses and modes of using which we do not find fully described by others. In fact, substitution groups constitute such a complex instrument that it seems almost impossible to gain a thorough knowledge of it from descriptions alone, especially since descriptions, as a rule, are unattractive to those who do not already possess kindred concepts which require only slight modification.

We have already remarked that the process of finding one substitution which is equivalent to two successive substitutions is called multiplication and that the commutative law does not hold in this multiplication. On this account it is necessary to distinguish between multiplier and multiplicand by the order in which they are written. On this point there is no uniformity among the writers on this subject but we shall always suppose that the multiplier precedes the multiplicand. Thus we have

$$abc \cdot ab = bc.$$

For in the first substitution a is replaced by b and in the second this b is replaced by a, thus a remains unchanged. In the first substitution b is replaced by c and in the second substitution this c is unchanged, hence the two successive substitutions replace b by c. In the first substitution c is replaced by a and in the second this a is replaced by b, hence the two successive substitutions replace c by b. Take for example the expression

$$a+3b+2c.$$

After applying the first substitution this becomes

$$b+3c+2a.$$

If we now operate with the second substitution this becomes

$$a+3c+2b.$$

which is the same result as we should have obtained by operating upon the first expression by bc.

In constructing groups we shall have to use multiplication to a very large extent. The process, is however, very simple as may be inferred from the preceding example and others which have been given before. The great importance of this operation led us to introduce this additional example with explanations. Referring to the lists of groups of two, three, and four letters, we desire to call attention to an important property. Let us consider, for instance the two groups $1,\ ab \cdot cd,\ ac \cdot bd,\ ad \cdot bc$

and $1,\ ac,\ bd,\ ac \cdot bd$

In the first one we observe that each letter is replaced by every other letter; while in the second a is replaced by c but not by either b or d, similarly d is replaced by b but not by a or c. Groups, like the first, in which one letter is replaced by every other letter of the group are called *transitive* groups. Those, like the second, in which no letter is replaced by all of the others are called *intransitive* groups. In the preceding lists there are only two intransitive groups but when the number of letters exceeds five this class of groups is by far the larger of the two.

The methods used to construct the intransitive groups are much simpler than those used to construct the transitive, on this account we shall consider the construction of the intransitive groups first.

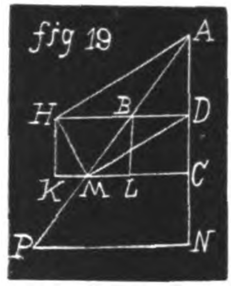
[To be continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton) Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the June Number.]

PROPOSITION XX. *Let there be a triangle* ACM (*fig. 19.*) *right-angled at* C. *Then from the point* B *bisecting this* AM *let fall the perpendicular* BD *to* AC. *I say this perpendicular will not be* (*in the hypothesis of acute angle*) *greater than half the perpendicular* MC.



Proof. For let DB be produced to DH double DB. Therefore DH would be (if DB be greater than the aforesaid half) greater than CM, and therefore equal to a certain continuation CMK.

Join AH, HK, HM, MD.

Now we proceed thus. Since in the triangles HBA, DBM, the sides HB, BA are assumed equal to the sides DB, BM; and (Eu. I. 15) the angles at the point B are equal; also (Eu. I. 4) the base HA will be equal to the base MD.

Then, by the same reasoning, in the triangles HBM, DBA, the bases HM, DA will be equal.

Wherefore in the triangles MHA, ADM, (Eu. I. 8) the angles MHA ADM will be equal. Again in the triangles AHB, MDB, the residual angle MHB will remain equal to the residual right angle ADB. Therefore the angle MHB will be right. But this is absurd in the hypothesis of acute angle; since the straight KH joining equal perpendiculars KC, HD, makes (P. III.) acute angles with these perpendiculars.

Therefore the perpendicular BD is not (in the hypothesis of acute angle) greater than the half of the perpendicular MC. Quod erat demonstrandum.

PROPOSITION XXI. *The same remaining; if* AM, *and* AC *are understood as produced in infinitum. I say their distance* (*in either hypothesis, of right angle or of acute angle*) *will be greater than any assignable finite length.*

Proof. In AM produced assume AP double AM, and let fall to AC produced the perpendicular PN.

The perpendicular MC will not be in either of the aforesaid hypothesis greater than half the perpendicular PN.

Therefore PN will be at least double MC, just as MC is at least double BD. And so always, if in AM produced is assumed double AP, and from the termination of this a perpendicular is let fall to AC produced.

It is obvious the perpendicular, which from AM ever more produced is let fall to AC produced, will be a multiple of the determinate BD beyond any finite assignable number.

Therefore the distance of the aforesaid straights will be (in either aforesaid hypotehsis) greater than any assignable finite length.

Quod erat demonstrandum.

THE INTRINSIC EQUATION OF A CURVE IN POLAR CO-ORDINATES.

By G. B. M. ZERR, A. M., Professor of Sciences, Texarkana, Texas.

The intrinsic equation of a curve has never been treated in polar co-ordinates in works on the subject. It is the aim of this paper to treat this subject briefly and to show that this method is somewhat simpler than the method of rectangular co-ordinates.

Let S denote the length of an arc of the curve $r=f(\theta)$ measured from some fixed point, φ the inclination of the tangent at the variable extremity to the tangent at the fixed point.

Let $y=f(x)$, then $\dfrac{dy}{dx}=f'(x)=-\cot\varphi \quad x=r\cos\theta, \quad y=r\sin\theta$

$$\frac{dx}{d\theta}=\cos\theta\frac{dr}{d\theta}-r\sin\theta,\ \frac{dy}{d\theta}=\sin\theta.\frac{dr}{d\theta}+r\cos\theta.$$

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{\sin\theta\frac{dr}{d\theta}+r\cos\theta}{\cos\theta\frac{dr}{d\theta}-r\sin\theta}=-\cos\varphi.$$

$$\therefore r\sin\theta\cot\varphi-\cos\theta\cot\varphi\frac{dr}{d\theta}=\sin\theta\frac{dr}{d\theta}+r\cos\theta.$$

$$\therefore \frac{dr}{d\varphi}=r\tan(\theta-\varphi)=f'(\theta)$$

$\therefore f'(\theta)=f(\theta)\tan(\theta-\varphi)\ldots.A$-

From A, θ is known in terms of φ, san $F(\varphi)$.

Then $\dfrac{d\theta}{d\varphi}=F'(\varphi)$.

$$\text{Now } \frac{ds}{d\theta}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}=\left[\left\{f(\theta)\right\}^2+\left\{f'(\theta)\right\}^2\right]^{\frac{1}{2}}$$

$$\therefore \frac{ds}{d\theta}=\{r^2+r^2\tan^2(\theta-\varphi)\}^{\frac{1}{2}}=r\sec(\theta-\varphi)=f(\theta)\sec(\theta-\varphi).$$

$$\frac{ds}{d\varphi}=\frac{ds}{d\theta}\cdot\frac{d\theta}{d\varphi}=F'(\varphi)f(\theta)\sec(\theta-\varphi)=rF'(\varphi)\sec(\theta-\varphi)\ldots.(B).$$

From B by substituting the value of r in terms of θ, and the value of θ, as found in A, in terms of φ, S may be found in terms of φ by integration.

I. Required the intrinsic equation of the circle.

The polar equation is $r=a$.

$$\therefore \frac{dr}{d\theta}=0=r\tan(\theta-\varphi).\quad \therefore \theta-\varphi=0, \text{ or } \theta=\varphi\ldots.(1).$$

$$\frac{d\theta}{d\varphi}=1,\ \therefore \frac{ds}{d\varphi}=a\sec(\varphi-\varphi)=a\sec 0=a.\quad \therefore S=a\varphi.$$

Or we may have proceeded thus:

$$\frac{ds}{d\theta}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}=a, \quad \therefore S=a\theta, \text{ but } \theta=\varphi, \quad \therefore S=a\varphi.$$

Taking the equation $r=2a\cos\theta\ \frac{dr}{d\theta}=-2a\sin\theta=r\tan(\theta=\varphi)$.

$$\therefore -2a\sin\theta=2a\cos\theta\tan(\theta-\varphi).$$

$$\therefore -\tan\theta=\tan(\theta-\varphi), \ \therefore \theta=\frac{\varphi}{2}\ldots\ldots(2).$$

$$\frac{d\theta}{d\varphi}=\tfrac{1}{2}. \quad \frac{ds}{d\varphi}=\tfrac{1}{2}\,.\,2a\cos\theta\sec(\theta-\varphi)=a\cos\frac{\varphi}{2}\sec\frac{\varphi}{2}=a,$$

$$\frac{ds}{d\varphi}=a,\ S=a\varphi, \quad \text{or } \frac{ds}{d\theta}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}=2a,\ S=2a\theta, \text{but } \theta=\frac{\varphi}{2}, \ \therefore S=a\varphi.$$

II. Required the intrinsic equation of the parabola. The polar equation is $r=\frac{2a}{1+\cos\theta}=\frac{a}{\cos^2\frac{\theta}{2}}$.

$$\frac{dr}{d\theta}=\frac{2a\sin\theta}{(1+\cos\theta)^2}=\frac{a\sin\frac{\theta}{2}}{\cos^3\frac{\theta}{2}}=r\tan(\theta-\varphi)=\frac{a\tan(\theta-\varphi)}{\cos^2\frac{\theta}{2}}.$$

$$\therefore \tan\frac{\theta}{2}=\tan(\theta-\varphi), \quad \therefore \theta=2\varphi\ldots\ldots(3).$$

$$\frac{d\theta}{d\varphi}=2. \quad \frac{ds}{d\varphi}=\frac{2a}{\cos^2\frac{\theta}{2}}\sec(\theta-\varphi)=\frac{2a}{\cos^3\varphi}.$$

$$\therefore S=\frac{a}{2}\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)+\frac{a\sin\varphi}{1-\sin^2\varphi} \quad \text{or thus}$$

$$\frac{ds}{d\theta}=\frac{a}{\cos^3\frac{\theta}{2}}. \quad \therefore S=\frac{a}{2}\log\left(\frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}\right)+\frac{a\sin\frac{\theta}{2}}{1-\sin^2\frac{\theta}{2}}, \text{ but } \theta=2\varphi.$$

$$\therefore S=\frac{a}{2}\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)+\frac{a\sin\varphi}{1-\sin^2\varphi}.$$

III. Find the intrinsic equation to the Cardioid. The polar equation is $r=a(1+\cos\theta)=2a\cos^2\frac{\theta}{2}$. $\frac{dr}{d\theta}=-2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$$=r\tan(\theta-\varphi)=2a\cos^2\frac{\theta}{2}\tan(\theta-\varphi).$$

$$\therefore -\tan\frac{\theta}{2}=\tan(\theta-\varphi), \ \therefore 2\varphi=3\theta, \ \frac{ds}{d\theta}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}$$

$$=2a\cos\frac{\theta}{2}, \quad \therefore\ S=4a\sin\frac{\theta}{2}=4a\sin\frac{\varphi}{3}.$$

IV. Find the intrinsic equation to the curve

$$r^{\frac{1}{m}}=a^{\frac{1}{m}}\cos\frac{\theta}{m} \text{ or } r=a\left(\cos\frac{\theta}{m}\right)^m. \quad \frac{dr}{d\theta}=-a\sin\frac{\theta}{m}\left(\cos\frac{\theta}{m}\right)^{m-1}$$

$$=r\tan(\theta-\varphi)=a\left(\cos\frac{\theta}{m}\right)^m\tan(\theta-\varphi).$$

$$\therefore -\tan\frac{\theta}{m}=\tan(\theta-\varphi). \quad \therefore\ (m+1)\theta=m\varphi. \quad \frac{ds}{d\theta}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}$$

$$=a\left\{\left(\cos\frac{\theta}{m}\right)^{2m}+\left(\sin\frac{\theta}{m}\right)^2\left(\cos\frac{\theta}{m}\right)^{2m-2}\right\}^{\frac{1}{2}}=a\left(\cos\frac{\theta}{m}\right)^{m-1}.$$

$$S=a\int\left(\cos\frac{\theta}{m}\right)^{m-1}d\theta=\frac{ma}{m+1}\int\left(\cos\frac{\varphi}{m+1}\right)^{m-1}d\varphi.$$

Let $n=\frac{1}{m}$, then $S=\frac{a}{n+1}\int\left(\cos\frac{n\theta}{n+1}\right)^{\frac{1}{n}-1}d\varphi$ is the intrinsic equation to the curve $r^n=a^n\cos n\theta$.

V. Find the intrinsic equation to the Logarithmic Spiral. The polar equation is $r=be^{\frac{\theta}{c}}$ or $r=ba^\theta$, $\log\frac{r}{b}=\frac{\theta}{c}$, $\frac{dr}{d\theta}=\frac{r}{c}=r\tan(\theta-\varphi)$.

$$\therefore \tan(\theta-\varphi)=\frac{1}{c}, \quad \theta=\tan^{-1}\frac{1}{c}+\varphi=d+\varphi.$$

$$\frac{ds}{d\theta}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}=\left(\frac{r^2d\theta^2+dr^2}{d\theta}\right)^{\frac{1}{2}}=\frac{\sqrt{1+c^2}\,dr}{d\theta}.$$

$$\therefore\ ds=\sqrt{1+c^2}\,dr, \quad S=\sqrt{1+c^2}\,.\,r=\sqrt{1+c^2}\,be^{\frac{\theta}{c}}=\sqrt{1+c^2}\,be^{\frac{d+\phi}{c}}.$$

The intrinsic equation for the evolute and involute can be found in the usual way.

A QUADRATIC CREMONA TRANSFORMATION DEFINED BY A CONIC.

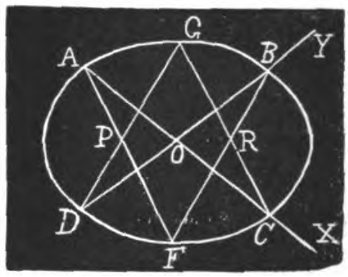
By LEONARD E. DICKSON, M. A., Fellow in The University of Chicago.

On an arbitrary conic choose four fixed points A, B, C, D. To each point P of the plane there corresponds a definite point R defined by the following construction: Join PA and PD cutting the conic again in F and G respectively. Then R is the intersection of BF with CG.

Let the equation of the base conic referred to the axes AC and BD have the general form $Ax^2+2Hxy+By^2+2Gx+2Fy+C=0 \ldots (1)$.

We may express the equation of the conic in terms of its four intercepts a, b, c, d on the axes and an unknown parameter. Since the points $A(-a, 0)$, $B(0, b)$, $C(c, 0)$, $D(0, -d)$ lie on the conic, we obtain the relations:

$$Aa^2-2Ga+C=0$$
$$Bb^2+2Fb+C=0$$
$$Ac^2+2Gc+C=0$$
$$Bd^2-2Fd+C=0.$$



From the first and third, $2G=A(a-c)$; from the second and fourth, $2F=-B(b-d)$. Hence $C=-Aac=-Bbd$. Substituting in (1); dividing by A, and writing $h=\frac{2H}{A}$,

$bdx^2+hbdxy+acy^2+bd(a-c)x+ac(d-b)y$ $-abcd=0 \ldots (2)$. The discriminant of (2), $\frac{-ac}{4b^2d^2}\{ab^2c+bc^2d+cd^2a+da^2b$ $+bdh(a-c)(b-d)-h^2b^2d^2\}$, will be 0 only when $h=\frac{ab+cd}{bd}$ or $h=\frac{-(ad+bc)}{bd}$.

In the former case, the conic (2) becomes $(bx+cy-bc)(dx+ay+ad)=0$, representing the straight lines AD and BC. We reject this trivial case, since to every point in the plane there corresponds the line BC. The quantity $ab+cd-hbd$ occurs below repeatedly.

For the case $h=\frac{-(ad+bc)}{bd}$, (2) becomes $(bx-ay+ab)(dx-cy-cd)=0$, representing the straight lines AB and DC. This case is not in the least trivial.

To find the co-ordinates x_1, y_1 of the point R, corresponding to a point $P(x', y')$.

The equation to AP is $y'(x+a)=y(x'+a)$; that to DP is $x(y'+d)$ $=x'(y+d)$. The co-ordinates of the second point of intersection F of AP with the conic (2) are:

$$\frac{c\{bd(x'+a)^2+ay'(b-d)(x'+a)-a^2y'^2\}}{bd(x'+a)^2+hbdy'(x'+a)+acy'^2},$$

$$\frac{y'\{ay'(bc-cd+hbd)+bd(a+c)(x'+a)\}}{bd(x'+a)^2+hbdy'(x'+a)+acy'^2}.$$

The co-ordinates of G, the second intersection of DP with the conic (2) are:
$$\frac{x'\{x'bd(c-a+dh)+ac(b+d)(y'+d)\}}{bdx'^2+bdhx'(y'+d)+ac(y'+d)^2},$$
$$\frac{-b\{d^2x'^2+dx'(a-c)(y'+d)+ac(y'+d)^2\}}{bdx'^2+bdhx'(y'+d)+ac(y'+d)^2}.$$

The equation to BF is
$$\frac{x}{y-b}=\frac{-c(ay'+dx'+ad)(ay'-bx'-ab)}{d\{bx'+(hb-c)y'+ab\}\{ay'-bx'-ab\}}=\frac{-c(ay'+dx'+ad)}{d\{bx'+(hb-c)y'+ab\}},$$
say $=\frac{A}{B}$.

The equation to CG is
$$\frac{x-c}{y}=\frac{\{dx'-cy'-cd\}\{b(a-dh)x'-acy'-acd\}}{b(dx'-cy'-cd)(dx'+ay'+ad)}$$
$$=\frac{b(a-dh)x'-acy'-acd}{b(dx'+ay'+ad)},\quad \text{say}=\frac{D}{E}.$$

By elimination, the co-ordinates of R are:
$$x_1=\frac{A(bD+cE)}{AE-BD},\quad y_1=\frac{E(bA+cB)}{AE-BD}.$$

Now $AE-BD$ expanded gives $-(ab+cd-bdh)(bdx'^2+acy'^2+bdhx'y'+abdx'+acdy')$; $bD+cE=bx'(ab+cd-bdh)$; $bA+cB=-cy'(ab+cd-bdh)$.
$$\therefore x_1=\frac{bcx'(dx'+ay'+ad)}{bdx'^2+acy'^2+bdhx'y'+abdx'+acdy'},$$
$$y_1=\frac{bcy'(dx'+ay'+ad)}{bdx'^2+acy'^2+\ldots\ldots}\ldots.(3).$$

Hence $\frac{x_1}{y_1}=\frac{x'}{y'}\ldots\ldots(4)$, or PR always passes through O, Pascal's Theorem for a hexagon inscribed in a conic.

Solving (3) for x' and y',
$$x'=\frac{adx_1(-bx_1-cy_1+bc)}{bdx_1^2+acy_1^2+bdhx_1y_1-abcy_1-bcdx_1},\quad y'=\frac{ady_1(-bx_1-cy_1+bc)}{bdx_1^2+\ldots.}\ldots.(5).$$

The reciprocity between P and R shown by (3) and (5) is evident geometrically.

If P describes a straight line $y=mx+l$, the locus of R is a conic.

Substituting the value (5) in $y'=mx'+l$, and dropping the subscripts to the co-ordinates of R, we find its locus: $bdx^2(l-ma)+dxy(ab-mac+blh)+acy^2(l+d)-abcy(l+d)-bcdx(l-ma)=0\ldots.(6)$.

Its discriminant is $-\frac{1}{4}ab^2c^2dl(l+d)(l-ma)(ab+cd-bdh)$. Rejecting as before the trivial case $h=\frac{ab+cd}{bd}$, this can be zero only if $l=0,-d$, or ma.

Hence (6) will degenerate to a pair of straight lines in just three cases:

If $l=0$, (6) becomes $ad(y-mx)(bx+cy-bc)=0$. Hence if P describes a straight line through O, the locus of R is this same line through O and the line BC, given when P is at O as the indeterminate intersection of BC with itself.

If $l=-d$, (6) becomes $-x\{bdx(ma+d)+dy(mac-ab+bdh)-bcd(ma+d)\}=0$. The second factor gives the equation to the line through C and the second intersection of $y=mx-d$ with the base conic (2). Hence the transform of any line DG through D is CG and the y-axis BD, the latter being given when P is at D as the intersection of BD with an indeterminate line through C.

If $l=ma$. (6) becomes $y\{adx(b-mc+mbh)+acy(d+ma)-abc(d+ma)\}=0$. The second factor gives the equation to the line through B and the second intersection of $y=m(x+a)$ with the base conic (2). Hence the transform of any line AF through A is BF and the x-axis AC, the latter being given when P is at A as the intersection of AC with an indeterminate line through B.

The conic (6) passes through the points O, B, C, and, since every point on the base conic is self corresponding, the points in which $y=mx+l$ intersects the base conic.

The line BC whose equation is $bx+cy-bc=0$ transforms into $b^2dx^2(a+c)+ac^2y^2(b+d)+bcdxy(2a+bh)-b^2cdx(a+c)-abc^2y(b+d)=0$ .....(7). The tangent to it at the origin, $bdx(a+c)+acy(b+d)=0$, passes through the intersection $\left(\frac{ac(b+d)}{ab-dc}, \frac{-bd(a+c)}{ab-dc}\right)$ of AD and BC. Further, (7) is tangent to the base conic (2) at the points B and C. Thus, the tangent at B to either (2) or (7) is $bdx(a-c+bh)+acy(b+d)-abc(b+d)=0$.

Applying (3), the equation to the curve which transforms *into* the line AD, or $dx+ay+ad=0$, is $bd^2x^2(a+c)+a^2cy^2(b+d)+abdxy(2c+dh)+abd^2x(a+c)+a^2cdy(b+d)=0$, which has the same tangent at the origin as (7).

The line at infinity transforms by (5) into the conic $bdx^2+acy^2+bdhxy-bcdx-abcy=0$....(8). Since (2) and (8) differ only by the linear expresssion $ad(bx+cy-bc)$, the points of intersection of BC with (2) lie on (8); also (2) and (8) are simultaneously ellipses, parabolas, or hyperbolas. The discriminant of (8), $-\frac{1}{4}ab^2c^2d(ab+cd-bdh)$, shows that it breaks up into two right lines only in trivial case above excluded.

The conic which transforms into the line at infinity, given by the vanishing of the denominator of (3), is $bdx^2+acy^2+bdhxy+abdx+acdy=0$....(9) passing through O, A, D. Subtracting (8) from (9), we find their intersections lie on the line $bdx(a+c)+acy(b+d)=0$, which passes through K, the intersection of AD and BC.

Note that the conics (2), (8) and (9) are *similar*

Generally, *the intersections of the conic which transforms into any straight line with the conic into which that straight line transforms lie two on the straight line OK and two on the line itself, the latter two being real and dis-*

tinct, coincident, or imaginary, according as the line intersects the base conic in two real, coincident, or imaginary points.

The conic which transforms into $y=mx+l$, given by substituting from (3) into $y_1=mx_1+l$, is $bdx^2(l+mc)+acy^2(l-b)+bxy(mac-cd+ldh)$ $+abdx(l+mc)+acdy(l-b)=0\ldots.(10)$.

This intersects the conic (6) into which $y=mx+l$ transforms in four points, which, if we subtract (6) from (10) and factor, are seen to lie on

$$\{acy(b+d)+bdx(a+c)\}\{mx-y+l\}=0.$$

O is one of the two intersections lying on OK. Call the other H. Then the point in which $y=mx+l$ meets OK and the point H mutually correspond. *We thus have an involution marked out on OK.*

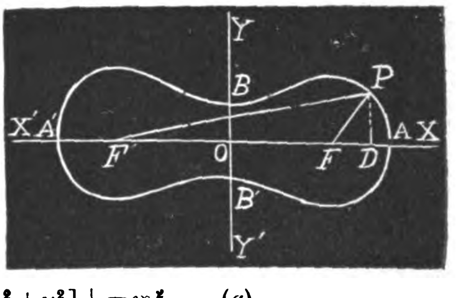
We saw above that the points A, D, O transform into the lines AC, DB, BC respectively. Now we can prove either geometrically or analytically that the lines AD, AO, DO transform into the points O, C, B respectively. Thus the sides and vertices of $\triangle ADO$ transform into the vertices and sides of $\triangle OBC$. With this exception the correspondence between the points in the two systems is one to one. The *projective* treatment of this transformation and its dual will be given elsewhere.

[For Projective Treatment, see my paper in the *Rendiconti del Circolo di Palermo.*]

THE RECTIFICATION OF THE CASSINIAN OVAL BY MEANS OF ELLIPTIC FUNCTIONS.

By F. P. MATZ, Sc. D., Ph. D., Mechanicsburg, Pennsylvania.

The Cassinian Oval is the locus of a point the *product* of whose distances from two fixed points is constant.



Let P be any point on the curve, F and F' the foci, O the middle point of FF', $OD=x$, $DP=y$, $OF=c$, $FP=\rho$, and $F'P=\rho'$; then, according to the definition of the curve, $\rho\rho'=m^2\ldots.(1)$. From the diagram, $\rho=\pm\sqrt{[(x-c)^2+y^2]}$ and $\rho'=\pm\sqrt{[(x+c)^2+y^2]}$; that is, from (1) we obtain the equation,

$$\sqrt{\{[(x-c)^2+y^2]\times[(x+c)^2+y^2]\}}=m^2\ldots.(a).$$

$\therefore\ (x^2+y^2+c^2)^2-4c^2x^2=m^4\ldots.(2)$; and this is the Cartesian equation of the Cassinian Oval, the co-ordinate axes being rectangular.

Put $OP=r$, and the $\angle POD=(\frac{1}{2}\pi-\theta)$; then $OD=x=r\sin\theta$, and $PD=y=r\cos\theta$. From (2), therefore, we have $r^4+2c^2(1-2\sin^2\theta)r^2=m^4-c^4\ldots.(b)$, or $r^4+(2c^2\cos2\theta)r^2=m^4-c^4\ldots.(3)$, which is a convenient form for the *central-polar* equation of the Cassinian Oval.

I. Rectification.—In order to *rectify* the Cassinian Oval, we deduce, from (3)

$$\cos 2\theta=\frac{(m^4-c^4)-r^4}{2c^2r^2}\ldots.(c).\quad \therefore \sin 2\theta=\sqrt{\left(\frac{4c^4r^4-[(m^4-c^4)-r^4]^2}{4c^4r^4}\right)}\ldots.(d).$$

$$\therefore \frac{d\theta}{dr}=\frac{(m^4-c^4)-r^4}{2c^2r^3\sin 2\theta},\ \text{and}\ \left(\frac{rd\theta}{dr}\right)^2=\frac{[(m^4-c^4)+r^4]^2}{4c^4r^4-[(m^4-c^4)-r^4]^2}\ldots.(e).$$

[The following transformation of (*c*) affords a *second* method for the derivation of (*e*):

$$\theta=\tfrac{1}{2}\cos^{-1}\left(\frac{(m^4-c^4)-r^4}{2c^2r^2}\right)=\cos^{-1}\left[\sqrt{\left(\frac{(m^4-c^4)-r^4}{4c^2r^2}+\tfrac{1}{2}\right)}\right]$$

$$=\cos^{-1}\left(\frac{\sqrt{[m^4-(c^2-r^2)^2]}}{2cr}\right),\ =\sin^{-1}\left(\frac{\sqrt{[(c^2+r^2)^2-m^4]}}{2cr}\right)\ldots.(f).]$$

From (*c*), when $\theta=0$, we have $r=\pm\sqrt{(m^2-c^2)}$, $=\pm b$; also, when $\theta=\frac{1}{2}\pi$, we have $r=\pm\sqrt{(m^2+c^2)}$, $=\pm a$. Since the perimeter of the Cassinian Oval is composed of four equal quadrantal arcs,

$$\mathsf{P}=4\int_b^a\sqrt{\left(\frac{4c^4r^4-[(m^4-c^4)-r^4]^2+[(m^4-c^4)+r^4]^2}{4c^4r^4-[(m^4-c^4)-r^4]^2}\right)}dr$$

$$=8m^2\int_b^a\frac{r^2dr}{\sqrt{\{[(c^2+r^2)^2-m^4]\times[m^4-(c^2-r^2)^2]\}}}=8m^2.$$

$$\int_b^a\frac{r^2dr}{\sqrt{\{[(m^2+c^2)+r^2]\times[(m^2+c^2)-r^2]\times[r^2+(m^2-c^2)]\times[r^2-(m^2-c^2)]\}}}$$

$$=8m^2\int_b^a\frac{r^2dr}{\sqrt{\{[(m^2+c^2)^2-r^4]\times[r^4-(m^2-c^2)^2]\}}}\ldots.(4).$$

Put $\quad r^4=(m^2+c^2)^2\cos^2\phi+(m^2-c^2)^2\sin^2\phi\ldots.(g);\quad$ then
$4r^3dr=2[-(m^2+c^2)^2+(m^2-c^2)^2]\sin\phi\cos\phi d\phi\ldots.(h).$

$$\therefore r^2dr=\frac{-2m^2c^2\sin\phi\cos\phi d\phi}{[(m^2+c^2)^2\cos^2\phi+(m^2-c^2)^2\sin^2\phi]^{\frac{1}{4}}}\ldots.(i).$$

Transforming (4) by means of (*g*) and (*i*), the expression for the required perimeter becomes

$$\mathsf{P}=8m^2\int_0^{\frac{1}{2}\pi}\frac{2m^2c^2[(m^2+c^2)\cos^2\phi+(m^2-c^2)^2\sin^2\phi]^{-\frac{1}{4}}\sin\phi\cos\phi d\phi}{\sqrt{\{[(m^2+c^2)^2-(m^2-c^2)^2]\sin^2\phi\times[(m^2+c^2)^2-(m^2-c^2)^2]\cos^2\phi\}}}$$

$$=4m^2\int_0^{\frac{1}{2}\pi}\frac{d\phi}{[(m^2+c^2)^2\cos^2\phi+(m^2-c^2)^2\sin^2\phi]^{\frac{1}{4}}}$$

$$=4m^2\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\{(m^2+c^2)^2-[(m^2+c^2)^2-(m^2-c^2)^2]\sin^2\phi\}^{\frac{1}{4}}}$$

$$=4m^2\int_0^{\frac{1}{2}\pi}\frac{d\phi}{[(m^2+c^2)^2-4m^2c^2\sin^2\phi]^{\frac{1}{4}}}=\frac{4m^2}{\sqrt{(m^2+c^2)}}\int_0^{\frac{1}{2}\pi}\frac{d\phi}{[1-C^2\sin^2\phi]^{\frac{1}{4}}}$$

$$= \frac{4m^2}{\sqrt{(m^2+c^2)}} \int_0^{\frac{1}{2}\pi} \left[1 + \frac{C^2 \sin^2 \phi}{4} + \frac{5C^4 \sin^4 \phi}{32} + \frac{15C^6 \sin^6 \phi}{128} + \text{etc.} \right] d\phi$$

$$= 2\pi \sqrt{\left(\frac{m^4}{m^2+c^2}\right)} \left[1 + \frac{C^2}{8} + \frac{15C^4}{256} + \frac{75C^6}{2048} + \text{etc.} \right] \ldots (5), \quad \text{in which}$$

$C^2 = 4m^2c^2 / (m^2+c^2)^2$. When $c=2$ and $m^4=25$, we deduce from (5) the numerical result, $P=12.7329+$, expressing the required perimeter. With respect to the *locus* represented by (3), we have the following hypotheses: $m>c\ldots(\alpha)$, $m=c\ldots(\beta)$, and $m<c\ldots(\gamma)$. Under the first hypothesis, the said locus represents the *Cassinian Oval*; under the second hypothesis, the said locus represents the *Bernoullian Lemniscate*; and under the third hypothesis, the said locus represents two *ovaliform figures*.

In rectifying under the hypotheses (β) and (γ), the term $(m^2-c^2)^2$ in (4) and in (g) must be altered accordingly.

[To be continued.]

THE CONSTRUCTION OF THE SUN'S PATH.

By ERIC DOOLITTLE, Professor of Mathematics in the State University of Iowa, Iowa City, Iowa.

In the *Archiv der Mathematik and Physik*, Vol. LIII. Part IV., there is a very interesting article by Professor Hoza on the graphical construction of the sun's apparent path. He considers the earth as stationary in its orbit during a perod of twenty-four hours, and obtains the projection of the apparent path during that time upon the plane of the Equator; the result being, as might be expected, an ellipse, the ratio of whose axes is as $1 : \cos\delta$.

This construction only applies to those places on the earth where the sun actually rises and sets each day, nor is the exact path thus found, since the sun's motion is not taken into account. An investigation of this latter is not difficult; it will lead us to a very interesting spiral curve.

Let us take the vernal equinox as an origin; the arc of the equator as an axis of X positive toward the right, and a great circle perpendicular to the equator through the vernal equinox as the axis of Y. The circles lie on the celestial sphere, whose radius is considered as unity.

Then, if c be the angular velocity of the earth on its axis, and K that of the sun in the ecliptic, (considered as uniform), we will have at a time t after the time of vernal equinox:

$$x = \tan^{-1}\left\{\cos e \tan Kt\right\} - ct \ldots .(1)$$

$$y = \sin^{-1}\left\{\sin e \sin Kt\right\} \ldots .(2)$$ where e, the obliquity of the ecliptic, is the constant angle 23° 27′.

Also, consider the point (x, y) as orthogonally projected upon the plane of the equator, and let the radius vector from the center of the sphere to the projected point be ρ. We may write the additional equation,

$$\rho^2 = \cos^2 y = (1 - \sin^2 e \sin^2 Kt) \ldots .(3).$$

By eliminating t between (1) and (2) we may obtain the equation of the path in spherial co-ordinates, or with (3) and the formula

$$\rho \sin \varphi = \sin x \ldots .(4),$$ we may find the equation of the projection upon the equator in polar co-ordinates. It will be easiest to examine the path by the help of (1), (2), and (3) without eliminating.

By differentiating (1)

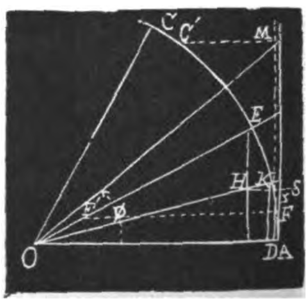
$$\frac{dx}{dt} = \frac{K \cos e}{\sin^2 e \sin^2 Kt + \cos^2 e} - c.$$

The maximum value of the first term of this expression occurs when $t = 0$: it is $K \sec e$ or, about $1.1K$. But $c = 366\frac{1}{4}K \ldots .(5)$ and hence $\frac{dx}{dt}$ is always minus, and x is always a decreasing function of t.

Similarly, $\frac{dy}{dt} = K \sin e \frac{\cos Kt}{\sqrt{1 - \sin^2 Kt \sin^2 e}}$ and y increases as Kt increases from 0 to $\frac{\pi}{2}$, at which point y attains the maximum value, e. But, by (5), as Kt increases to $\frac{\pi}{2}$, ct increases to $183\frac{1}{8}\pi$, and, from (1), x decreases to $-182\frac{5}{8}\pi$. Thus the spiral begins at the vernal equinox, and reaches its highest point after $91\frac{5}{16}$ revolutions about the sphere.

When Kt increases to π, y decreases to 0, and x decreases to $-365\frac{1}{4}\pi$, so that the spiral turns downward and crosses the equator after $182\frac{5}{8}$ revolutions. The negative values are similar and show that the spiral begins to repeat itself after $365\frac{1}{4}$ revolutions.

Equation (3) shows us that the projection on the plane of the equator is a similar spiral, which begins at the vernal equinox and draws continually nearer the pole for 91 revolutions, after which it widens out until $\rho = 1$ at the autumnal equinox. The minumum value of ρ is $\cos e$, and the curve is always concave toward the pole.



We may make an accurate construction of this projection by points, as follows:

Draw the unit circle and lay off the given angles and their sines:

$$AOK = Kt; \quad AOE = e; \quad AOC = ct.$$

Say $OF' = \sin Kt$ and draw $F'F$ and FD.

Then $AOF = y$ (by (2)) and $OD = \cos y = \rho$.

Draw HS, meeting the tangent in z. Then $AOZ =$ first term of x and if AZ' be laid backward from C to C', $AOC'=x$. Finally, draw $C'M$ to DF; then $AOM=\varphi$ by (4), and AOM and OD are co-ordinates of points on our spiral. In the figure, AK was taken 10°, and thus $AC=20\pi+62\frac{1}{2}^\circ$.

So accurate a construction might be used sometimes; for instance for the time of sun-rise and sun-set especially near the poles, that is where the spiral cuts the projected Horizon circle. For showing the form of the spiral it can of course be much shortened: $\tan^{-1}(\cos e \tan Kt)$ may be written Kt, and in fact, when it is noticed how very nearly parallel the spires are, many convolutions may be interpolated without computation.

A METHOD OF INTEGRATING CERTAIN DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND N-TH DEGREE.

By RALPH H. KUNSTADTER, Graduate Student, Yale College.

The regular solution of the equation $e^{ax} f(p) \pm e^{by} f'(p)=0$, (where $p=\frac{dy}{dx}$) as given in text-books on differential equations is performed after substituting x' far e^{ax} and y' for e^{by}.

In the following demonstration, I shall endeavor to give a very rational and perhaps a natural method which might be easily generalized for similar equations consisting of two members only.

Regarding our equation, we at once recognize it as being a trancendental and exponential equation, which in order to solve we will bring into logarithmic form. For this purpose, multiply $e^{ax}f(p)+e^{by}f'(p)=0\ldots.(1)$ by $e^{ax}f(p)-e^{by}f'(p)=0$ and we have $e^{2ax}f^2(p)-e^{2by}f'^2(p)=0\ldots.(2)$.

Applying logarithms, we have $\log e^{2ax}+2\log f(p)=\log e^{2by}+2\log f'(p)\ldots.(3)$. Or $ax+\log f(p)=by+\log f'(p)\ldots.(4)$.

Differentiating this equation, we have

$$adx+\frac{\varphi(p)dp}{f(p)}=bdy+\frac{\varphi'(p)dp}{f'(p)}\ldots.(5).$$

Dividing by dx and clearing of fractions,

$$f(p)f'(p)a+[\varphi(p)f'(p)-f(p)\varphi'(p)]\frac{dp}{dx}+bpf'(p)f(p)=0\ldots.(6).$$ Or

$$\frac{dp}{dx}[\varphi(p)f'(p)-f(p)\varphi'(p)]+af(p)f'(p)+bpf'(p)f(p)=0.$$

We see that the separation of $\psi(p)dp$ and of dx can be distinctly performed and hence the solution of our problem is theoretically done. Should the given equation be of the form $e^{ax}f(p)-e^{by}f'(p)=0$, we obtain the logarithm of it without multiplying by a factor.

To illustrate our method take the equation,

$$e^{3x}(p-1)+e^{2y}p^3=0; \quad \text{or} \quad e^{3x}\left(\frac{dy}{dx}-1\right)+e^{2y}\left(\frac{dy}{dx}\right)^3=0.$$

We have $e^{6x}(p-1)^2-e^{4y}(p)^6=0$. Applying logarithms and dividing by 2, $3x+2\log(p-1)-2y-3\log p=0$.

Differentiating this equation, we have

$$3dx+\frac{dp}{p-1}-2dy-3\frac{dp}{p}=0, \quad \text{or} \quad 3p^2dx-3pdx+pdp-2p^2dy+2pdy-3pdp$$

$+6dp=0$. Dividing through by dx, we get $\frac{dp}{dx}(3-2p)+\frac{dy}{dx}(2p-2p^2)+3p^2$

$-3p=0$, $\frac{dy}{dx}(3-2p)=2p^3+3p-5p^2$, and $\int\frac{dp(3-2p)}{2p^3+3p-5p^2}=x$.

It is not necessary to continue this as it is now to be treated in the customary way.

Similarly, we solve $e^{3x}\left(\frac{dy}{dx}-1\right)-e^{2y}\left(\frac{dy}{dx}\right)^3=0$.

ARITHMETIC.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

48. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Fifty thousand days preceding Thursday, March 7, 1895, was what date and what day of the week?

I. Solution by P. S. BERG, Apple Creek, Ohio.

Every four years previous to March 7, 1895, excepting the four years of which 1800 was one, contained 1461 days. This number is contained in 50000, 34 times with a remainder of 326 days. Since 1800 was not a leap year the 34 periods or 136 years conducts back to March 6th 1759. 326 days further leads to April 14th 1758.

By referring to a table in Olmsted's Astronomy I find this date to have occurred on Thursday.

II. **Solution by S. HART WRIGHT, Ph. D., Penn Yan, New York.**

Dividing 50000 by 7 gives 6 remainder, and six days before Thursday falls on Friday, the day of the week required.

Any four consecutive years, containing one bissextile year have 1461 days. $50000 \div 1461$ gives 34 four-year periods, hence there are 34—1 bissextile days, the year 1800 not being a leap-year. $50000-33=49967$ days and 49967 $\div 365$, gives 136 years +327 days. (1895+66 days)—(136 years +327 days) gives 1758+104 days= April 14, 1758 the required date, in Gregorian Calender or April 3 in the Julian Calender.

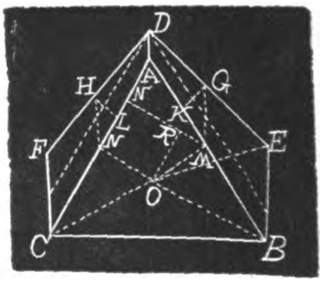
A. L. FOOTE gets as his result Thursday April 13, 1758.

49. **Proposed by J. A. CALDERHEAD, B Sc., Superintendent of Schools, Limaville, Ohio.**

I have a garden in the form of an equilateral triangle whose sides are 200 feet. At each corner stands a tower; the height of the first tower is 30 feet, the second 40 feet and the third 50 feet. At what distance from the base of each tower must a ladder be placed, that it may just reach the top of each? And what is the length of the ladder, the garden being a horizontal plane?

Solution by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Construction.—Let ABC be the triangular garden and AD, BE, and CF the towers at the corners. Connect the tops of the towers by the lines ED and DF. From G and H, the middle points of DE and DF, draw GM and HN perpendicular to DE and DF, and at M and N draw perdendiculars to AB and AC in the triangle ABC, meeting at O. Then O is equally distant from D and E. For, since M is equally distant from D and E, and MO perpendicular to the plane $ABED$, every point of MO is equally distant from D and E. For a like reason, every point of NO is equally distant from D and F; hence, O their point of intersection, is equally distant from D, E, and F and is, therefore, the point where the ladder must be placed. Draw DI and DJ parallel to AB and AC, GK and HL perpendicular to AB and AC, MP perpendicular to AC and OR parallel to NP. Draw the lines OB, OC, and OA, the required distances from the base of the ladder to the bases of the towers. Draw EO, the length of the ladder.



1. $AB=BC=AC=200$ ft. $=s$, the side of the triangle.
2. $FC=50$ ft. $=a$, the height of the first tower,
3. $EB=40$ ft. $=b$, the height of the second tower, and
4. $AD=30$ ft. $=c$, the height of the third tower. Let
5. $h=\sqrt{[AB^2-(\frac{1}{2}AC)^2]}=\sqrt{[s^2-(\frac{1}{2}s)^2]}=\frac{1}{2}\sqrt{3}s=100\sqrt{3}$ ft. =the perpendicular from B to the side AC.
6. $EI=BE-BI(=AD)=(b-c)=40$ ft. -30 ft. $=10$ ft.
7. $GK=\frac{1}{2}(EB+AD)=\frac{1}{2}(b+c)=\frac{1}{2}(40$ ft. $+30$ ft.$)=35$ ft. In the similar triangles DIE and GKM,
8. $DI:IE::GK:KM$, or $s:b-c::\frac{1}{2}(b+c):KM$.

9. $\therefore\ KM = \frac{b^2 - c^2}{2s} = \frac{40^2 - 30^2}{2 \times 200} = 1\frac{3}{4}$ ft.,

10. $AM = AK + KM = \frac{1}{2}s + \frac{b^2 - c^2}{2s} = \frac{s^2 + b^2 - c^2}{2s} = 101\frac{3}{4}$ ft., and

11. $BM = AB - AM = s - \frac{s^2 + b^2 - c^2}{2s} = \frac{s^2 + c^2 - b^2}{2s} = 98\frac{1}{4}$ ft.

In like manner,

12. $HL = \frac{1}{2}(a + c) = \frac{1}{2}(50 \text{ ft.} + 30 \text{ ft.}) = 40$ ft.,

13. $LN = \frac{a^2 - c^2}{2s} = 4$ ft.,

14. $AN = AL + LN = \frac{1}{2}s + \frac{a^2 - c^2}{2s} = \frac{s^2 + a^2 - c^2}{2s} = 104$ ft.

15. $NC = AC - AN = s - \frac{s^2 + a^2 - c^2}{2s} = \frac{s^2 + c^2 - a^2}{2s} = 96$ ft.

By similar triangles,

16. $AB : AL :: AM : AP$, or $s : \frac{1}{2}s :: (s^2 + b^2 - c^2) \div 2s : AP$.

Whence,

17. $AP = (s^2 + b^2 - c^2) \div 4s = 50\frac{7}{8}$ ft.

A.

18. $\therefore\ PL = AL - AP = [\frac{1}{2}s - (s^2 + b^2 - c^2) \div 4s] = (s^2 + c^2 - b^2) \div 4s = 49\frac{1}{8}$ ft.

19. $RO = PN = PL + LN = (s^2 + c^2 - b^2) \div 4s + (a^2 - c^2) \div 2s = (s^2 + 2a^2 - b^2 - c^2) \div 4s = 53\frac{1}{8}$ ft. By similar triangles,

20. $AB : BL :: AM : MP$, or $s : \frac{1}{2}\sqrt{3}s :: (s^2 + b^2 - c^2) \div 2s : MP$.

Whence,

21. $MP = [(s^2 + b^2 - c^2) \div 4s] \times \sqrt{3} = 50\frac{7}{8}\sqrt{3}$ ft. By similar triangles,

22. $MP : AP :: RO : RM$, or $[(s^2 + b^2 - c^2) \div 4s]\sqrt{3} : (s^2 + b^2 - c^2) \div 4s :: (s^2 + 2a^2 - b^2 - c^2) \div 4s : RM$.

23. $RM = (s^2 + 2a^2 - b^2 - c^2)4\sqrt{3}s = [(s^2 + 2a^2 - b^2 - c^2) \div 12s]\sqrt{3} = 17\frac{17}{24}\sqrt{3}$ ft. Again

24. $MP : MA :: RO : OM$, or $[(s^2 + b^2 - c^2) \div 4s]\sqrt{3} : (s^2 + b^2 - c^2) \div 2s :: (s^2 + 2a^2 - b^2 - c^2) \div 4s : OM$.

25. $\therefore\ OM = (s^2 + 2a^2 - b^2 - c^2) \div 2\sqrt{3}s = [(s^2 + 2a^2 - b^2 - c^2) \div 6s]\sqrt{3} = 35\frac{5}{12}\sqrt{3}$ ft.

26. $ON = RP = MP - RM = [(s^2 + b^2 - c^2) \div 4s]\sqrt{3} - (s^2 + 2a^2 - b^2 - c^2) \div 12s]\sqrt{3} = [(s^2 - a^2 + 2b^2 - c^2) \div 6s]\sqrt{3} = 33\frac{1}{6}\sqrt{3}$ ft.

Then

II.

27. $OC = \sqrt{(ON^2 + NC^2)} = \sqrt{\left[\left(\frac{s^2 - a^2 + 2b^2 - c^2}{6s}\sqrt{3}\right)^2 + \left(\frac{s^2 + c^2 - a^2}{2s}\right)^2\right]} = \sqrt{[(33\frac{1}{6}\sqrt{3})^2 + 96^2]} = \sqrt{12516\frac{1}{12}}$ $= 111.8796+$ ft.

28. $OA = \sqrt{(ON^2 + AN^2)} = \sqrt{\left[\left(\frac{s^2 - a^2 + 2b^2 - c^2}{6s}\sqrt{3}\right)^2 + \left(\frac{s^2 + c^2 - a^2}{2s}\right)^2\right]} = \sqrt{[(33\frac{1}{6}\sqrt{3})^2 + 104^2]} = \sqrt{14116\frac{1}{12}}$ $= 118.8111+$ ft.

29. $OB = \sqrt{(OM^2 + MB^2)} = \sqrt{\left[\left(\frac{s^2 + 2a^2 - b^2 - c^2}{6s}\sqrt{3}\right)^2 + \left(\frac{s^2 + c^2 - b^2}{2s}\right)^2\right]} = \sqrt{[(35\frac{5}{12}\sqrt{3})^2 + (98\frac{1}{4})^2]}$

$=\frac{1}{4}\sqrt{214657\frac{1}{3}}=115.8278+$ ft.

1. B. $OE=\sqrt{(BE^2+OB^2)}=\sqrt{[(\frac{1}{4}\sqrt{214657\frac{1}{3}})^2+40^2]}$,
$=\sqrt{(13416\frac{1}{3}+1600)}=\sqrt{15016\frac{1}{12}}=122.5402+$ ft. = the length of the ladder.

III. ∴
1. 111.8796+ft. = the distance from base of the ladder to the base of the tower FC,
2. 118.8111+ft. = the distance from the base of the ladder to the base of the tower AD.
3. 115.8278+ft. = the distance from the base of the ladder to the base of the tower BE, and
4. 122.5402+ft. = the length of the ladder.

[From *Finkel's Mathematical Solution Book*, *p.* 299.]

[NOTE.—This method of solution may be easily extended to the more general case, viz., when the triangle is scalene.]

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

44. Proposed by LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago.

Find the general term in the series 1, 3, 10, 35, 126, 462, 1716, 6435, 24310, ..., which plays a remarkable part in some recent theorems in my Theory of Regular Polygons.

Solution by the PROPOSER.

This series is a "diagonal" in the Triangle of Pascal, as shown in the following table:—

C	0	1	2	3	4	5	6	7	8	
0	1									
1	1	**1**								
2	1	2	1							
3	1	3	**3**	1						
4	1	4	6	4	1					
5	1	5	10	**10**	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	**35**	21	7	1		
8	1	8	28	56	70	56	28	8	1	
9	1	9	36	84	126	**126**	84	36	9	1

Since the *mth* term in the series lies at the intersection of column m with row $(2m-1)$, it is $\dfrac{(2m-1)(2m-2)\ldots.(2m-m)}{1.2.3\ldots.m}$

$$=\frac{(2m-1)(2m-2)\ldots.(m+1)}{1.2.3\ldots.(m-1)}=\frac{(2m)!}{2(m\,!)^2}.$$

The part played by the series in the Theory of Regular Polygons is indicated in an article in the current issue of the *Annals of Mathematics*.

Also solved by *A. H. BELL*, and *Professor J. F. W. SCHEFFER*.

45. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.**

Find x from $\cos^{-1}\dfrac{1-x^2}{1+x^2}+\tan^{-1}\dfrac{2x}{1-x^2}=\dfrac{4\pi}{3}$.

I. **Solution by T. W. PALMER, A. M., Professor of Mathematics, University of Alabama; H. W. DRAUGHON, Olio, Mississippi; A. L. FOOTE, C. E., Middlebury, Connecticut; and the PROPOSER.**

Let $\tan\theta=x$. Then $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}, =\dfrac{2x}{1-x^2}$; and $\dfrac{1-x^2}{1+x^2}$

$$=\frac{1-\tan^2\theta}{1+\tan^2\theta}, =\cos^2\theta(1-\tan^2\theta), =(\cos^2\theta-\sin^2\theta), =\cos 2\theta.$$

$$\therefore \cos^{-1}\frac{1-x^2}{1+x^2}+\tan^{-1}\frac{2x}{1-x^2}, =\cos^{-1}\cos 2\theta+\tan^{-1}\tan 2\theta=\frac{4\pi}{3}.$$

$$\therefore 2\theta+2\theta, =4\theta, =\frac{4\pi}{3}, \quad \therefore \theta=\frac{\pi}{3}, =60^\circ.$$

$$\therefore x=\tan\theta, =\tan\frac{\pi}{3}, =\tan 60^\circ, =\sqrt{3}.$$

$$\therefore x=\sqrt{3}.$$

II. **Solution by JOHN B. FAUGHT, A. B., Indiana University, Bloomington, Indiana; J. A. JOHNSON, Jr., Student of the Sophomore Class, University of Mississippi; P. S. BERG, Apple Creek, Ohio; and J. W. WATSON, Middle Creek, Ohio.**

$$\text{Since } \tan^{-1}\frac{2x}{1-x^2}=\cos^{-1}\frac{1}{\sqrt{1+\dfrac{4x^2}{(1-x^2)^2}}}=\cos^{-1}\frac{1-x^2}{1+x^2};$$

$$\therefore 2\cos^{-1}\frac{1-x^2}{1+x^2}=\frac{4}{3}\pi, \text{ or } \cos^{-1}\frac{1-x^2}{1+x^2}=\frac{2}{3}\pi=\cos^{-1}(-\tfrac{1}{2}).$$

$$\therefore \frac{1-x^2}{1+x^2}=-\tfrac{1}{2}, \text{ whence } x^2=3, \text{ and } x=\pm\sqrt{3}.$$

Also solved by *F. P. MATZ, J. SCHEFFER, C. D. SCHMITT, E. L. SHERWOOD, M. C. STEVENS, G. B. M. ZERR;* and ————

PROBLEMS.

54. **Proposed by E. W. MORRELL, Department of Mathematics, Montpelier Seminary, Montpelier, Vermont.**

Transform $x^4+y^4+z^4-2y^2z^2-2z^2x^2-2x^2y^2$— into a product.
[*Bowser's Trigonometry.*]

55. **Proposed by MARCUS BAKER, M. A., U. S. Geological Survey, Washington, D. C.**

Two right triangles ABC and ABD are so placed as to have one side $x(=AB)$ in common. From P the intersection of their hypotenuses is drawn c perpendicular to x. Knowing the hypotenuses $a=39$ feet and $b=25$ feet and the perpendicular $c=12\frac{6}{7}$ feet, find x. Note this theorem $\frac{1}{m}+\frac{1}{n}=\frac{1}{c}$ or $\frac{1}{\sqrt{a^2-x^2}}+\frac{1}{\sqrt{b^2-x^2}}=\frac{1}{c}$, where m and n are the altitudes of the two triangles, respectively. Also find locus of P. Discuss the case when the triangles are general (not right angled.)

[The same problem, in the form of "two poles" with ropes stretched from top of one to foot of other and the same data given, was contributed by H. C. Wilkes. Ed.]

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

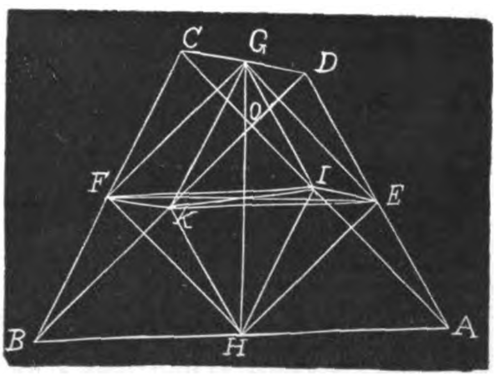
SOLUTIONS OF PROBLEMS.

43. **Proposed by J. F. W. SCHEFFER, Hagerstown, Maryland.**

The consecutive sides of a quadrilateral are a, b, c, d. Supposing its diagonals to be equal, find them and also the area of the quadrilateral.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics, Inter State College, Texarkana, Texas.

Let $AB=a$, $BC=b$, $CD=c$, $DA=d$, $AC=DB=x$, $EF=y$, $GH=z$; also, let H, F, G, E, I, K be the middle points of AB, BC, CD, DA, AC, DB, respectively.



Then $EGFH$, $GKHI$, $FKEI$ are all parallelograms; but $IG=HK=\frac{1}{2}d$, $HI=KG=\frac{1}{2}b$, $EI=KF=\frac{1}{2}c$, $FI=EK=\frac{1}{2}a$, $EG=GF=FH=HE=\frac{1}{2}x$.

$\therefore \frac{1}{2}(a^2+c^2)=y^2+$

$IK^2 \ldots.(1)$, $\frac{1}{2}(b^2+d^2)=z^2+IK^2 \ldots.(2)$, $x^2=y^2+z^2 \ldots.(3)$.

$(1)-(2)$ gives, $\frac{1}{2}(a^2+c^2-b^2-d^2)=y^2-z^2 \ldots.(4)$.

From (3) and (4), $y=z$, and $a^2+c^2=b^2+d^2 \ldots.(5)$,

$(a^2-b^2)^2=(c^2-d^2)^2 \ldots.(6)$.

$\therefore$ $EGFH$ is a square and AC is perpendicular to BD.

$\therefore$ area $ABCD=\frac{1}{2}x^2$.

$\frac{1}{2}x^2=\frac{1}{2}bx \sin ACB+\frac{1}{2}cx \sin ACD$

$$\sin ACB=\frac{\sqrt{4b^2x^2-(x^2+b^2-a^2)^2}}{2bx},\quad \sin ACD=\frac{\sqrt{4c^2x^2-(x^2+c^2-d^2)^2}}{2cx}.$$

$$\therefore 2x^2=\sqrt{4b^2x^2-(x^2+b^2-a^2)^2}+\sqrt{4c^2x^2-(x^2+c^2-d^2)^2} \ldots.(7).$$

From (5), (6), and (7) $2x^4-2x^2(a^2+c^2)+(a^2-d^2)^2+(c^2-d^2)^2=0$.

$$\therefore x^2=\tfrac{1}{2}\left\{a^2+c^2\pm\sqrt{(a^2+c^2)^2-2(a^2-d^2)^2-2(c^2-d^2)^2}\right\}.$$

If $a=b=c=d$, $x^2=2a^2$.

This problem was also solved by *F. P. MATZ*, *J. C. CORBIN*, and *J. F. W. SCHEFFER*.

44. Proposed by I. J. SCHWATT, Ph. D., Professor of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania.

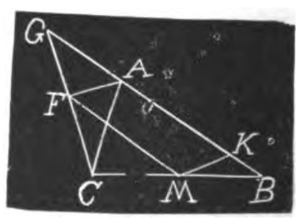
(1). If from the middle point M of the side BC of the triangle ABC a parallel to the external bisector AF of the angel BAC is drawn to meet AB at K, the point K divides then the side AB in KA

$=\frac{1}{2}(AB+AC)$ and $KB=\frac{1}{2}(AB-AC)$.

(2). If K is joined to the extremity D of the diameter perpendicular to BC then is KD perpendicular to AB.

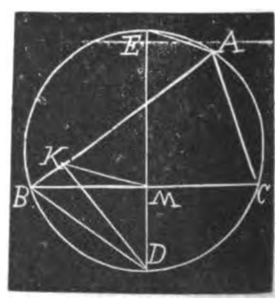
Solution by the PROPOSER.

(1.) If from the middle point M of the side BC of the triangle ABC a parallel to the external bisector AF of the angle BAC is drawn to meet AB at K then is the side AB divided in $KA=\frac{1}{2}(AB+AC)$ and KB $=\frac{1}{2}(AB-AC)$.



From M draw a parallel to AB intersecting the external bisector of BAC at F. Join C to F and produce CF to meet AB at G. Since $CM=MB$ and MF is parallel to AB, therefore is $CF=FG$; The triangles CAF and FAG are equal and therefore is $AG=AC$. $MF=AK=\frac{1}{2}(AB+AG)=\frac{1}{2}(AB+BC)$. $BK=AB-\frac{1}{2}(AB+BC)=\frac{1}{2}(AB-BC.)$

(2.) If K is joined to the extremity D of the diameter perpendicular to BC then is KD perpendicular to AB.



The internal and external bisectors of an angle are perpendicular to each other. Since AD is the internal bisector of the angle BAC therefore is EA the external bisector of the same angle. By (1) is MK parallel to AE or angle EAB=angle AKM; but angle EAB equals angle EDB therefore is angle

AKM equal to angle EDB or the points B, D, M and K are concyclic and therefore is angle BMD=angle $BKD=90°$.

This problem was also solved by *G. B. M ZERR, O. W. ANTHONY, E. W. MORRELL*, and *P. S. BERG*.

PROBLEMS.

48. Proposed by I. J. SCHWATT, Ph. D., Professor of Mathematics in the University of Pennsylvania, Philadelphia, Pennsylvania.

The Simson line belonging to one point of intersection of Brocord's Diameter, of a triangle with the circumcircle of this triangle is either parallel or perpendicular to the bisector of the angle formed by the side BC of the triangle ABC and the corresponding side $B'C'$ of Brocard's triangle.

49. Proposed by J. C. WILLIAMS, Rome, New York.

Of all triangles inscribed in a given segment of a circle, with the chord as base, the isosceles is the maximum.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

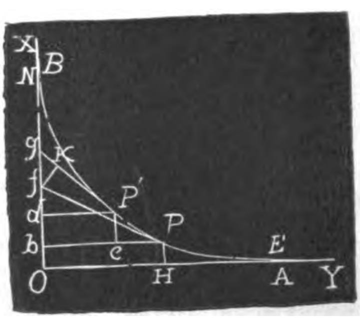
SOLUTIONS OF PROBLEMS.

34. Proposed by GEORGE LILLEY, Ph. D., LL. D., Park School, Portland, Oregon.

A hare is at O, and a hound at E, 40 rods east of O. They start at the same instant each running with uniform velocity. The hare runs north. The hound runs directly toward the hare and overtakes it at N, 320 rods from O. How far did the hound run?

I. Solution by M. C. STEVENS, A. M., Professor of Mathematics, Purdue University, Lafayette, Indiana; COOPER D. SCHMITT, A. M., University of Tennessee, Knoxville, Tennessee; and G. B. M. ZERR, A. M., Ph. D., Inter States College, Texarkana, Texas.

Let u= velocity of hound, v= velocity of hare, $AO=a$, $BO=b$,



$OH=x$, $PH=y$, $AP=s$, $\frac{v}{u}=n$. Then $nPA=$ Of. $\therefore ns=y-x\frac{dy}{dx}$.

Differentiating with respect to x, we have $n\frac{ds}{dx}=-x\frac{d^2y}{dx^2}$. But s increases as x diminishes, whence $\frac{ds}{dx}=-\sqrt{1+\left(\frac{dy}{dx}\right)^2}$. $\therefore \frac{n}{x}$

$= \dfrac{\dfrac{d^2y}{dx^2}}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}$. Integrating and noting that $x=a$, $\dfrac{dy}{dx}=0$, together, n

$\log\dfrac{x}{a}=\log\left[\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}+\dfrac{dy}{dx}\right]$. $\therefore \left(\dfrac{x}{a}\right)^n=\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}+\dfrac{dy}{dx}$, also by reciprocals, $\left(\dfrac{a}{x}\right)^n=\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}-\dfrac{dy}{dx}$. Addding we get $2\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}=2\dfrac{ds}{dx}$

$=\left(\dfrac{x}{a}\right)^n+\left(\dfrac{a}{x}\right)^n$. $\therefore s=\frac{1}{2}\int_0^a\left[\left(\dfrac{x}{a}\right)^n+\left(\dfrac{a}{x}\right)^n\right]dx=\frac{1}{2}\left[\dfrac{x^{n+1}}{a^n(n+1)}+\dfrac{a^nx^{1-n}}{1-n}\right]_0^a$

$=\dfrac{a}{1-n^2}$. $\therefore s=\dfrac{a}{1-n^2}$, but $ns=b$, $\therefore n=\dfrac{b}{s}$. $\therefore s=\dfrac{as^2}{s^2-b^2}$, $\therefore s^2-as=b$,

$s=\frac{1}{2}[a\pm\sqrt{(a^2+4b^2)}]$ $s=\frac{1}{2}[a+\sqrt{(a^2+4b^2)}]$, the minus sign not being admissable. $s=\frac{1}{2}[40+\sqrt{(1600+409600)}]=340.624+$rods. To find the rectangular equation to the curve, we have by subtracting $2\dfrac{dy}{dx}=\left(\dfrac{x}{a}\right)^n-\left(\dfrac{a}{x}\right)^n$. $\therefore 2(y+C)$

$=\dfrac{x^{n+1}}{a^n(n+1)}-\dfrac{a^nx^{1-n}}{1-n}$. When $y=0$, $x=a$, which gives $C=-\dfrac{an}{1-n^2}$.

$$\therefore 2\left(y-\frac{an}{1-n^2}\right)=\frac{x^{n+1}}{a^n(n+1)}-\frac{a^nx^{1-n}}{1-n}.$$

II. **Solution by ALFRED HUME, C. E., D. Sc., University of Mississippi, and the PROPOSER.**

The tangent to the path of the hound always passes through the position of the hare, the point of tangency being the simultaneous position of the hound. Let P and P' be two positions of the hound, PP' being infinitesimal, f and g corresponding positions of the hare. Draw the ordinates $P'd$ and Pb and the perpendiculars fk and $P'e$ to $P'g$ and Pb respectively. Let $AP=s$, $PP'=ds$, $OB=x$, $bd=eP'=dx$, $OF=z$, $fg=dz$. Now, since the two animals run with uniform velocities, $\dfrac{ds}{dz}=$ some constant$=l$. In the limit, $\triangle$'s $P'eP$ and fgk are similar.

$$\therefore \frac{ds}{dx}=\frac{dz}{gk};\quad gk=dx\frac{dz}{ds}=\frac{dx}{l}.$$

The change in the length of the tangent as the hound runs from P to P' is due to a positive increment, gk, at one end, and a negative increment, PP', at the other. Therefore, if $t=$ the length of the tangent, $dt=\dfrac{dx}{l}-ds$, and, integrating, $t=\dfrac{x}{l}-s+C$. When $t=40$, $x=0$, and $s=0$; $\therefore C=40$. For

s substitute its value lz; then $t=\frac{x}{l}-lz+40$. When $t=0$, $x=320$, and $z=320$.

$$\therefore 0=\frac{320}{l}-320l+40, \text{ and } l=1.0644+. \quad s=lz=1.0644\times320=340.$$

624+, the distance in rods that the hound runs.

Also solved by *A. L. FOOTE*, *P. S. BERG* and *F. P. MATZ*.

35. **Proposed by H. C. WHITAKER, B. S., C. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.**

At the rate of 10 cubic inches per second, water is running into a vessel in the shape of a right conic frustum, the radii of whose upper and lower bases are respectively 15 and 10 inches, and whose altitude is 20 inches. At what rate per second is the *depth* of the water increasing, when *it* is exactly 8 inches?

I. **Solution by J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering, A. and M. College, College Station, Texas, and ALFRED HUME, C. E., D. Sc., University of Mississippi.**

Represent the depth and the volume of water at any instant by h and V respectively. The radius of the free surface of the water will be $10+\frac{h}{4}$.

Therefore $V=\frac{\pi}{3}h\left[100+\left(10+\frac{h}{4}\right)^2+10\left(10+\frac{h}{4}\right)\right]$.

Differentiating, $dV=\frac{\pi}{3}(300+15h+\frac{3}{16}h^2)$. Substituting for dV and h 10 and 8 respectively, $dh=\frac{5}{72\pi}$, or the depth is increasing at the rate of $\frac{5}{72\pi}=0.02210484+$ of an inch per second.

[This result may also be obtained as follows: The area of the free surface of the water at the instant under consideration is 144π. Therefore the depth is increasing at the rate of $\frac{10}{144\pi}$ inches per second.]

II. **Solution by Professor J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.**

Let the radii of upper and lower bases be a and b, the altitude$=h$, $x=$ the distance of water from the lower base after the time t, y the radius of section at the distance x; then, denoting by c the quantity of water flowing in per second, $\pi y^2dx=cdt$.

$$\therefore \frac{dx}{dt}=\frac{c}{\pi y^2}=\frac{ch^2}{\pi[(a-b)x+bh]^2}.$$

For the given numerical values, we have $\frac{dx}{dt}=\frac{5}{72\pi}$=required rate at distance $=8$.

Also solved by *P. S. BERG*, *F. P. MATZ* and *E. L. SHERWOOD*.

PROBLEMS.

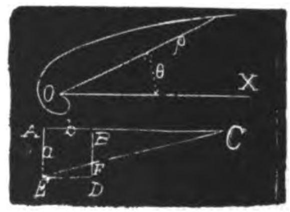
43. Proposed by Professor C. E. WHITE, A. M., Trafalgar, Indiana.

Prove that $\int_0^1 \frac{x^{a-1}+x^{-a}}{1+x}\cdot\frac{dx}{x}=\log\left(\tan\frac{a\pi}{2}\right)$, when $a>0$ and <1.

[*Williamson's Int. Calculus, p. 154.*]

44. Proposed by DE VOLSON WOOD, C. E., Professor of Mechanical and Electrical Engineering in Stevens Institute of Technology, Hoboken, New Jersey.

Find the equation of a curve in which $\rho=f(\theta)$, in which ρ is equal to BC, an intercept of any secant drawn from the corner E of the rectangle $AEDB$, and prolonged to cut AB prolonged in C. Let equal increments of θ be proportional to the equal increments of DB as divided by the secant EF, θ being zero when EC coincides with ED, and $\theta=2\pi$ when EF passes through B. Determine the asymptotes.



[Prof. Mac Cord of Stevens Institute, while investigating the curve of intersection of a plane with the surface of a certain volume, found that it had the property of the above problem, and he referred it to Professor Wood to investigate in regard to asymptotes. If the curve is not known to science, Professor Mac Cord desires to christen it as "The Thistle of Scotland."]

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

23. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University Post Office, Mississippi.

A heavy particle is placed upon the surface $\sqrt{x}+\sqrt{y}+\sqrt{z}=c$. The axis of z being vertical and the coefficient of friction being $\sqrt{2}$, show that a point of equilibrium (all friction possible being brought into action) z is a harmonical mean between x and y.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Member of London Mathematical Society and Professor of Mathematics, Ohio University, Athens, Ohio.

If X, Y, Z be the forces of which P is the resultant independent of the reaction of the surface on the particle parallel to the co-ordinate axes, $\mu=\sqrt{2}=$ the coefficient of friction, and $\mu=0$ the equation to the surface, for limiting equilibrium we have

$$\left(X\frac{du}{dx}+Y\frac{du}{dy}+Z\frac{du}{dz}\right)^2\div P^2\left(\frac{du^2}{dx^2}+\frac{du^2}{dy^2}+\frac{du^2}{dz^2}\right)=\frac{1}{1+\mu^2}\ldots\ldots(1),$$

the derivatives being partial.

We have $u=\sqrt{x}+\sqrt{y}+\sqrt{z}-c=0\ldots\ldots(2)$, whence $\frac{du}{dx}=\frac{1}{2\sqrt{x}},\ \frac{du}{dy}=\frac{1}{2\sqrt{y}},\ \frac{du}{dx}=\frac{1}{2\sqrt{z}}\ \ldots.(3).$

Also, $X=Y=0,\quad Z=-g=P\ldots.(4).$

Substituting in (1) and reducing, $z=\frac{2xy}{x+y}\ldots.(5).$

II. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi.

If W is the weight of the particle, N and T its normal and tangential components,

$$W^2=N^2+T^2.$$

Also, when the particle is on the point of sliding,

$$T=\sqrt{2}\,N.$$

Hence $W^2=3N^2$.

Again $W\cos\theta=N$, θ being the angle between the normal and the Z-axis.

Now $\cos\theta=\dfrac{\frac{dF}{dz}}{\sqrt{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}\right)^2}}$, $F(x,y,z)=0$ being the equation of the surface, and the differential-coefficients being partial.

$$\frac{dF}{dx}=\frac{1}{2\sqrt{x}},\quad \frac{dF}{dy}=\frac{1}{2\sqrt{y}},\quad \frac{dF}{dz}=\frac{1}{2\sqrt{z}};$$

and, therefore, $\cos\theta=\dfrac{\frac{1}{\sqrt{z}}}{\sqrt{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}}$.

But $W^2\cos^2\theta=N^2=\frac{W^2}{3}$, from which $3\cos^2\theta=1$.

Substituting, $\frac{3}{z}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$, or $\frac{2}{z}=\frac{1}{x}+\frac{1}{y}$, and z is a harmonical mean between x and y.

Solutions of this problem were also received from *F. P. MATZ* and *G. B. M. ZERR*.

24. Proposed by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

A sphere whose center of gravity does not coincide with its geometrical center is placed on a rough inclined plane. State under what circumstances the sphere will slide without rolling, roll without sliding, and neither roll nor slide.

Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics, Irving College, Mechanicsburg, Pennsylvania.

On a horizontal plane, the sphere will neither roll nor slide; but it will *rock* about the vertical drawn through the point of support. Down an inclined plane, the sphere will roll without sliding, until the *initial horizontal plane* through which the centroid has (by the rolling) become a *vertical* plane. So long as this initial horizontal plane remains a vertical plane, the sphere will slide without rolling.

25. Proposed by Professor GEORGE LILLEY, LL. D., Ex-President of Washington State Agricultural College and School of Science, Portland, Oregon.

It is known that if the velocity of a certain freight train is 30 miles an hour it can be brought to a stand still in a distance of 500 feet by setting the brakes. It was stopped in 1200 feet by setting the brakes. Find its velocity, the forces exerted by the brakes being the same in each case.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Physics, Inter State College, Texarkana, Texas, and the PROPOSER.

$\frac{1}{2}Mv^2=Rs$, where M=mass, v=velocity, R=resistance of brakes, s=distance train runs after setting brakes.

30 miles per hour=44 feet per second.

$\therefore \frac{1}{2}M(44)^2=500R\ldots.(1)$. $\frac{1}{2}Mv^2=1200R\ldots.(2)$.

$$(2)\div(1),\quad \left(\frac{v}{44}\right)^2=\frac{12}{5}. \quad \therefore 5v^2=12(44)^2.$$

$\therefore v=88\sqrt{\frac{3}{5}}$ feet per second $=60\sqrt{\frac{3}{5}}$ $=46.4758$ miles per hour.

Also solved by *F. P. MATZ* and *E. W. MORRELL.*

PROBLEMS.

31. Proposed by O. W. ANTHONY, Mexico, Mo.

A perfectly elastic but perfectly rough sphere of mass M and radius R, rotating in a vertical plane with an angular velocity of ω, is let fall from a height, a, upon a perfectly elastic but perfectly rough horizontal plane. Determine the motion of the body after striking the plane. What will be its ultimate motion?

32. Proposed by OTTO CLAYTON, A. B., Fowler, Indiana.

The wheel of a wind pump has 60 fans, each turned at an angle 45° to the direction of the axis, and each having 150 square inches exposed to the wind. If the wind blows with velocity V and the wheel rotates with velocity ω what is the component of force or pressure along the axis if it is turned at an angle α to the direction of the wind assuming the resistance the wheel meets in turning to be R?

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

25. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find, if possible, integral values of each of the seven linear measurments of a rectangular parallolopiped; i. e. length, breadth, height, the diagonals of each of the three different rectangular sides, and the diagonal from an upper corner to the opposite lower corner; or, find integral values, if possible, of a, b, c, d, e, f, and g, as shown in the equations,— $a^2+b^2=c^2$, $a^2+d^2=e^2$; $a^2+f^2=g^2$, $b^2+d^2=f^2$, $b^2+e^2=g^2$, $c^2+d^2=g^2$, $c^2+e^2=f^2$. If not possible, how many of them can have integral values? and which?

Solution by G. B. M ZERR. A. M., Ph. D., Vice-President and Professor of Mathematics and Sciences, Inter State College, Texarkana, Texas.

Let the length, breadth, and height be, $a=8mn(m^4-n^4)$; $b=2mn\{10m^2n^2-3(m^4+n^4)\}$, $c=(m^2-n^2)(m^4+n^4-14m^2n^2)$.

The method of obtaining the above values has been published in several journals and need not be repeated here. From the above we easily get $a^2+b^2=\{2mn(5m^4-6m^2n^2+5n^4)\}^2$, $a^2+c^2=(m^6+17m^4n^2-17m^2n^4-n^6)^2$, $b^2+c^2=(m^6+3m^4n^2+3m^2n^4+n^6)^2=(m^2+n^2)^6$, and $a^2+b^2+c^2=64m^2n^2(m^4-n^4)^2+(m^2+n^2)^6$. This last is a square when $64m^2n^2(m^2-n^2)^2+(m^2+n^2)^4=\square$. Let $m=pn$. Then must $64p^2(p^2-1)^2+(p^2+1)^4=p^8+68p^6-122p^4+68p^2+1=\square$.

I have not yet succeeded in making this last a square. The edges and diagonals of sides, are integral, satisfying six of the relations.

Let $m=2$, $n=1$; then $a=240$, $b=44$, $c=117$, $\sqrt{a^2+b^2}=244$, $\sqrt{a^2+c^2}=267$, $\sqrt{b^2+c^2}=125$, $\sqrt{a^2+b^2+c^2}=5\sqrt{2929}$.

26. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find (1) a *square fraction* the arithmetical difference of whose terms is a *cube*: and (2) find a *cubic fraction* the arithmetical sum of whose terms is a *square*.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

(1) Let $\frac{y^2}{x^2}$ equal the *square fraction*. Then $x^2-y^2=a$ *cube*$=a^3$. Then $(x+y)(x-y)=a^3$. Put $x+y=a^2$, and $x-y=a$.

$$\text{Then } x=\frac{a^2+a}{2}=\frac{a(a+1)}{2}; \text{ and } y=\frac{a^2-a}{2}=\frac{a(a-1)}{2}.$$

Whatever integral values be assigned to a, x and y will always be integral; for whether a is even or odd, $a(a+1)$ and $a(a-1)$ are *even*.

$$\text{Since } x-y=a,\ \frac{a(a-1)}{2}+a=\frac{a(a+1)}{2}.$$

$\therefore$ the denominator of the fraction $\frac{y}{x}$ is always a more than the numerator. Also, as $\frac{a(a-1)}{2}$ is the sum of the series $(1+2+3+\ldots\ldots+a-1)$, $\frac{y}{x}=(1+2+3+\ldots\ldots+a-1)\div[(1+2+3+\ldots\ldots+a-1)+a]$, or $(1+2+3+\ldots\ldots+a-1)\div(1+2+3+\ldots\ldots+a)$. Putting a equal, consecutively, to the successive integers beginning with *unity*, we have, respectively, $\frac{y}{x}=\frac{0}{1}, \frac{1}{3}, \frac{3}{6}, \frac{6}{10}, \frac{15}{21}, \frac{21}{28}, \frac{28}{36}, \frac{36}{45}$, etc., *ad infinitum*.

(2) Let $\frac{m^3}{n^3}$ be the *cubic fraction*. Then $n^3+m^3=\square=b^2$. Then $(n+m)(n^2-nm+m^2)=b^2$. Put $n+m=n^2-nm+m^2=b$.

$$\text{Then } n=\tfrac{1}{2}\left(b+\sqrt{\frac{b(4-b)}{3}}\right), \text{ and } m=\tfrac{1}{2}\left(b-\sqrt{\frac{b(4-b)}{3}}\right).$$

The only integral values of b that will render $\sqrt{\frac{b(4-b)}{3}}$ rational, are 3 and 4. Whence the respective values of n are 2 and 2, and those of m are 1 and 2. $\therefore \frac{m}{n}=\frac{1}{2}$ or $\frac{2}{2}$.

By putting $n+m=\frac{b}{2}$, and $n^2-nm+m^2=2b$ we obtain $\frac{m}{n}=\frac{0}{4}$ and $\frac{4}{8}$. Other relations of the factors, both in (1) and (2), will yield other results.

II. **Solution by E. L. SHERWOOD, A. M., Professor of Mathematics and Science, Mississippi Normal College, Houston, Mississippi.**

1st. Let $(x+1)^2=$ one of the terms, and $(a-x)^2$ the other. Then $(a-x)^2-(x+1)^2=$a cube$=(x+1)^3$. Finding a in terms of x, we get, $a=x+(x+1)\sqrt{(x+2)}$, and substituting in the first equation, $[(x+1)\sqrt{(x-2)}]^2-(x+1)^2=(x+1)^3$. Now substitute n^2-2 for x and the last equation becomes $(n^2-1)^2n^2-(n^2-1)^2=(n^2-1)^3$.

$\therefore$ Fraction$=\frac{n^2(n^2-1)^2}{(n^2-1)^2}$, or $\frac{(n^2-1)^2}{n^2(n^2-1)^2}$; difference$=(n^2-1)^3$, in which n may be any integer.

2nd. Since both terms must be cubes, we must have

$$n^3(n^3+1)^3+(n^3+1)^3=\text{a square}=(n^3+1)(n^3+1)^3.$$

$\therefore$ Fraction$=\frac{n^3(n^3+1)^3}{(n^3+1)^3}$, or $\frac{(n^3+1)^3}{n^3(n^3+1)^3}$; Sum$=(n^3+1)^4$, in which n may be any integer

III. **Solution by H. C. WILKES, Murrayville, West Virginia, and A. H. BELL, Hillsboro, Illinois.**

Since when the first number is unity, the sum of any number of successive cubes is a square, if we let $n=$ a root of any cube, then

$$\left[n\left(\frac{n+1}{2}\right)\right]^2-\left[n\left(\frac{n-1}{2}\right)\right]^2=n^3.$$

Let $n=2$, then $3^2-1^2=2^3$,
" $n=3$, then $6^2-3^2=3^3$,
" $n=4$, then $10^2-6^2=4^3$, or in the series 1, 3, 6, 10, 15, etc., we have the difference of the squares of any two contiguous terms equal a cube.

Second case. Let $\frac{y^3}{x^3}$ be the fraction. Then $x^3+y^3=n^2$, or $(x+y)(x^2-xy+y^2)=n^2$. If $(x+y)$ be a square, then x^2-xy+y^2 will be a square. This is only possible when $x=y$. $\therefore$ the sum of any two equal cubes, the sum of whose roots is a square, will be a square, as $\frac{8^3}{8^3}$, or $\frac{512}{512}$, will be an improper cubic fraction the sum of whose terms will be a square. If $x+y=x^2-xy+y^2$, then x^3+y^3 will be a square. This is only possible when $x=2$, $y=1$, and the proper fraction $\frac{1}{8}$ will be cubic and the sum of the terms a square number.
$\therefore (2^3+1^3)=3^2$; also $2^6(2^3+1^3)=24^2$, etc.

Also solved by *P. S. BERG*, *A. L. FOOTE*, *G. B. M. ZERR*, and the *PROPOSER*.

PROBLEMS.

34. Proposed by R H. YOUNG, West Sunbury, Pennsylvania.

Prove (1) that $\frac{n(n+1)(2n+1)}{6}$ is a whole number for all values of n; and (2) prove that $\frac{(n-1)n(n+1)}{24}$ is a whole number when n is odd.

35. Proposed by G. B. M. ZERR, A. M., Ph. D., Vice-President and Professor of Mathematics and Sciences, Inter State College, Texarkana, Texas.

Decompose into the sum of two squares the number $13^2.61^3$.

36. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find the first six integral values of n in $\frac{n(n+1)}{2}=\square$.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

19. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Find the average area of the circle which is the locus of the middle points of all chords passing through a point taken at random in the surface of a given circle.

II. Solution by JOHN DOLMAN, Jr., Philadelphia, Pennsylvania.

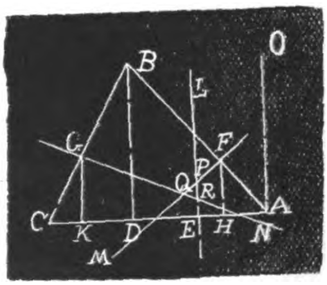
If the random point be taken upon the circumference of a variable circle, concentric with the given circle, the area of the circular "locus" is evidently always one quarter of this variable circle. Again: As the random points are equably distributed over the area of the given circle, the variable circle must always increase in area by equal increments, and, as it varies from zero to the full area of the given circle, its mean area is, of course, half of the given circle. Therefore the mean area of the locus named is one half of one quarter or one eighth of the given circle.

23. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Find the average area of all the triangles that can be drawn *perpendicular-sided* to a given plane scalene triangle.

Solution by G. B. M. ZERR, A. M., Ph. D., Vice-President and Professor of Mathematics and Sciences, Inter State College, Texarkana, Texas.

Let ABC, be the given scalene triangle; EL, FM, GN the perpendiculars to AC, AB, BC respectively, forming the triangle PQR. Let AC, AO be the axes of co-ordinates. Draw BD, FH, GK perpendicular to AC.



Let $AD=d$, $BC=e$, $BD=h$, $AC=b$, $AB=c$, $BC=a$, $AE=u$, $AH=v$, $AK=w$.

Then $FH=hv/d$, $GK=\dfrac{h(b-w)}{e}$. $x=u$,

$$y=\frac{hv}{d}+\frac{dv}{h}-\frac{dx}{h},\quad y=\frac{h(b-w)}{e}+\frac{(b-d)(x-w)}{h},$$

are the equations to LE, FM, and GN respectively.

$$\therefore\ \frac{h^2ev+d^2ev+w(h^2d-d^2e+bde)-h^2db}{bde}=\frac{c^2ev+a^2dw-h^2db}{bde},$$ is the abscissa of Q. Let A_1=area PQR, $\triangle$= area ABC.

$$\therefore\ A_1=\frac{\triangle\,(c^2ev+a^2dw-h^2db-bdeu)^2}{b^2d^2e^2h^2}=\tfrac{1}{2}\,\frac{(c^2ev+a^2dw-h^2db-bdeu)^2}{bd^2e^2h}.$$

The limits of u are 0 and b; of v, 0 and d; of w, d and b.

$$\therefore\ A=\frac{\int_0^b\int_0^d\int_d^b A_1\,du\,dv\,dw}{\int_0^b\int_0^d\int_d^b du\,dv\,dw}=\frac{1}{bde}\int_0^b\int_0^d\int_d^b A_1\,du\,dv\,dw$$

$$=\frac{1}{6a^2b^2d^4e^3h}\int_0^b\int_0^d\left\{(c^2ev+e^2db-bdeu)^3-(c^2ev-h^2bd+a^2d^2-bdeu)^3\right\}du\,dv$$

$$=\frac{1}{24a^2b^2c^2e^4h}\int_0^b\left\{(c^2e+e^2b-beu)^4-(e^2b-beu)^4-(c^2e-h^2b+a^2d-beu)^4\right.$$

$$+(a^2d-h^2b-beu)^4\Big\}du$$

$$=\frac{1}{60a^2b^3c^2h}\Big\{(a^2+bd)^5-(a^2-be)^5-b^5e^5-b^5d^5\Big\}$$

$$=\frac{1}{120\triangle a^2b^2c^2}\Big\{\Big(\frac{a^2+b^2+c^2}{2}\Big)^5-\Big(\frac{a^2+c^2-b^2}{2}\Big)^5-\Big(\frac{a^2+b^2-c^2}{2}\Big)^5-\Big(\frac{b^2+c^2-a^2}{2}\Big)^5\Big\}$$

$$=\frac{4\triangle^4}{15a^2b^2c^2}\Big\{(\cot A+\cot B+\cot C)^5-\cot^5 A-\cot^5 B-\cot^5 C\Big\}$$

$$=\frac{a^4+b^4+c^4}{48\triangle}.$$

The last four expressions are the same and show the beautiful relations existing between the terms of the triangle.

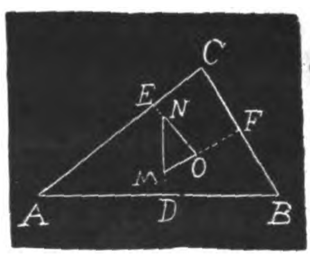
This problem was also solved by *F. P. MATZ*. His solution is published in this issue as solution of problem 24, that problem being identical with 23

24. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

The average area of the triangle formed by three perpendiculars drawn from the sides of the triangle (a, b, c), is $\mathbf{A}=(a^4+b^4+c^4)/48\triangle$.

Solution by the PROPOSER.

Let $BC=a$, $CA=b$, $AB=c$, $BF=x$, $CE=y$, and $AD=z$. The $\triangle MNO$ is similar to the $\triangle ABC$; and it may be *wholly within*, *partly within*, or *wholly without*, the $\triangle ABC$. Put $\angle NAD=\phi$, and $\angle MBD=\theta$; then $AN=(b-y)/\cos(A-\phi)$ $=z\cos\phi\ldots(\alpha)$, and $ND=z\tan\phi\ldots(\beta)$. By means of (α), we have from (β), $ND=[(b-y)-z\cos A]/\sin A\ldots(1)$. Also, $BM=(c-z)/\cos(B-\theta)=x/\cos\theta\ldots(\gamma)$, and $MD=(c-z)\tan(B-\theta)\ldots(\delta)$. By means of (γ) we have from (δ), $MD=[x-(c-z)\cos B]/\sin B\ldots(2)$. Subtracting (2) from (1), we have the expression:



$$MN=\frac{b\sin B+c\sin A\cos B-(x\sin A+y\sin B+z\sin C)}{\sin A\sin B}\ldots(3).$$

The *first two* terms in the numerator of (3) may be symmetrically written thus: $a\sin B\cos C+b\sin C\cos A+c\sin A\cos B,=\frac{1}{2}(a\sin A+b\sin B+c\sin C)\ldots(\epsilon)$. Transforming (3) by means of (ϵ), we have

$$MN=\frac{(a-2x)\sin A+(b-2y)\sin B+(c-2z)\sin C}{2\sin A\sin B}\ldots(4).$$

Representing the numerator of the right-hand member of (4) by $\mathbf{N}$, we have (by symmetry) the expressions:

$$NO=\mathbf{N}/2\sin C\sin A,\text{ and }OM=\mathbf{N}/2\sin B\sin C;$$

and, consequently, the $\triangle MNO=\frac{1}{8}(\mathbf{N})^2 / \sin A \sin B \sin C \ldots (5)$. The expression for the average area of the $\triangle MNO$, therefore, becomes

$$\mathbf{A}=\frac{1}{8abc \sin A \sin B \sin C}\int_0^a\int_0^b\int_0^c(\mathbf{N})^2 dxdydz$$

$$=\tfrac{1}{24}\left(\frac{a^2 \sin A}{\sin B \sin C}+\frac{b^2 \sin B}{\sin C \sin A}+\frac{c^2 \sin C}{\sin A \sin B}\right)=\frac{a^4+b^4+c^4}{48\triangle}.$$

[Note.—Problems twenty-three and twenty-four are identical. This fact was not observed until after they were both printed. Ed.]

PROBLEMS.

31. **Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.**

Find the average length of a line drawn at random across the opposite sides of a rectangle whose length is l and breadth b.

32. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.**

Find the average area of the random sector whose vertex is a random point in a given circle.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

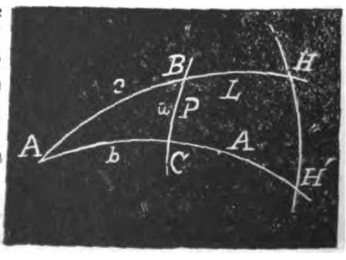
SOLUTIONS OF PROBLEMS.

17. **Proposed by SAMUEL HART WRIGHT, M. D., M. A., Ph. D. Penn Yan, New York.**

A bright star passed my meridian at 7 P. M. The Chronometer soon after ran down and stopped, but I set it again when the same star had a true altitude of $30^\circ=\alpha$. What time was it then, my latitude being $42^\circ\ 30'$ N. $=\lambda$, and the star's declination 60° N. $=\delta$

II. Solution by the PROPOSER.

Let B be the north pole, A the zenith, C the star, HH' the horizon, AH and AH' each $=90^\circ$, AH' being a meridian, AH a verticle circle, BH' the altitude of the pole = the latitude $=L$, $AB=$co-latitude $=c$, $BC=a=$ polar distance of the star$=P$, $AC=b=$the zenith distance of star, $CH=A=$altitude of the star, and the angle $ABC=$the hour-angle of the star$=T$ in siderial time. Put $s=\frac{1}{2}(a+b+c)$, and $s-a=a'$, $s-b=b'$, and $s-c=c'$. Then by *Sph. Trig.*



$$\sin\tfrac{1}{2}T=\sqrt{\left(\frac{\sin c' \sin a'}{\sin c \sin a}\right)}, \text{ and}$$

$\frac{1}{2}T=51^\circ 40' 18''.5$....(1) or $\cos\frac{1}{2}T=\sqrt{[\sin s \sin(s-b) \text{cosec } c \text{ cosec } a]}$, and $\frac{1}{2}T=51^\circ 40' 18''.5$....(2), or in terms of A, L, and P; put $s'=\frac{1}{2}(A+L+P)$, then $s=\frac{1}{2}(180^\circ-A-L+P)=90^\circ-s'+P=90^\circ-A-L+s'$, and $s-P=90^\circ-s'$, and $s-(90^\circ-L)=s'-A$. Whence, $\sin\frac{1}{2}T=\sqrt{[\sec L \text{ cosec } P \cos s' \sin(s'-A)]}$, and $\frac{1}{2}T=51^\circ 40' 18''.5$ ...(3).

$\therefore$ $T=103^\circ 20' 37''=6$ hr. 53 min. 22.467 sec. of siderial time=6 hr. 52 min. 14.75 sec. mean solar time. To this add 7 hrs., the time the star was on the meridian, and we get 1 hr. 52 min. 14.75 sec. of the morning of the next day, for the time when the chronometer was set.

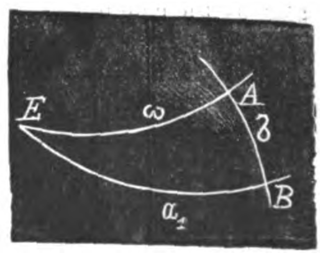
In Bowditch's *Practical Navigator*, pp. 209-210, the rules for finding T are translations of eqs. (2) and (3), but no reasons for the rules are given, and no formulas from which they are derived. The above formulas, (1), (2), and (3) are as applicable for obtaining correct time on land as at sea. [This solution is important as showing how the Rule in Bowditch's *Navigator* is obtained,—which some very good mathematicians have failed to comprehend.—EDITOR.]

20. Proposed by SAMUEL HART WRIGHT, M. D., M. A., Ph. D., Penn Yan, New York.

When does the Dog-Star and the Sun rise together in latitude $42^\circ 30'$ N. $=\lambda$, given the R. A. of Sirius=6 hrs. 40 min. 30 sec., and its Dec. $=16^\circ 33' 56''$ S.?

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science in Inter State College, Texarkana, Texas.

Let $\lambda=$ latitude of observer, $\alpha=$R. A., $\delta=$declination, $t=$hour angle of Sirius. $\alpha_1=$R. A., $\delta_1=$declination, $t_1=$hour angle of sun, $\varepsilon=$obliquity of the ecliptic, $\omega=$ distance from vernal equinox to the sun's position, $\tau=$ time of sun-rise before six o'clock. Then we get $\cos t=-\tan\lambda\tan\delta$....(1). $\cos t=-\tan\lambda\tan\delta_1$....(2). $\sin\alpha_1=\tan\delta_1 \cot\varepsilon$....(3). $\alpha_1-t_1=\alpha-t=\theta$, or $\alpha_1=\theta+t_1$ ....(4). $\sin\alpha_1=\sin(\theta+t_1)$....(5). $\cos\varepsilon=\cot\omega \tan\alpha_1$....(6). $\sin\tau=\tan\lambda\tan\delta_1$....(7). From (3) and (5), $\sin(\theta+t_1)=\tan\delta_1\cot\varepsilon$....(8). From (2) and (8),



$$\tan\delta_1=\frac{\sin(\theta+t_1)}{\cot\varepsilon}=-\frac{\cos t_1}{\tan\lambda}\text{....(9).}$$

From (9), $$\tan t_1=-\frac{\cot\varepsilon+\sin\theta\tan\lambda}{\cos\theta\tan\lambda}\text{....(10).}$$

But $\lambda=42^\circ 30'$, $\alpha=6$ hr. 40 min. 30 sec., $\delta=16^\circ 33' 56''$ S., $\varepsilon=23^\circ 27' 13''$.

From (1), $t=74^\circ 10' 57''.81=4$ h. 56 m. 43.85 sec.

$\alpha-t=\theta=1$ h. 43 m. 46.15 sec. $=25^\circ 56' 32''.25=$siderial time when Sirius rises.

From (10), $t_1=107^\circ 3' 20''=7$h. 8 m. 13.33 sec.

From (4), $\alpha_1=t_1+\theta=8$ h. 51 m. 59.48 sec. $=132^\circ 59' 52''.25$.

From (3), $\delta_1=17^\circ 36' 16''.85$. From (6), $\omega=130^\circ 32' 38''.5$.

From (7), $\tau=16^\circ\ 54'\ 12''.41=1$ h. 7 m. 36.83 sec. before six o'clock.

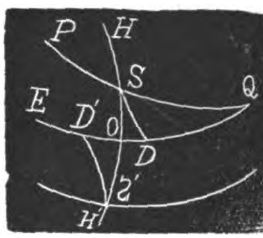
$\therefore$ The sun rises 4 h. 52 m. 23.17 sec. after mid-night. It takes the sun 186 days to go from the vernal to the autumnal equinoxes.

$\therefore\ 180^\circ : 186=130^\circ\ 32'\ 33''.5 : 134.8955$.

March 20+134.8955=August 2. $\therefore$ The event takes place August 2, 4 h. 52 m. 23.17 sec. To find the time when the sun and Sirius set together we have $\theta=a+t=a_1+t_1=11$ h. 37 m. 13.85 sec. The rest of the calculation is the same as that given above.

III. Solution by the PROPOSER.

Let HH' be the horizon, EQ the equator, PQ the ecliptic, S the sun at rising, S' Sirius at rising, $D'S'$ the Dec.$=\delta$, $D'Q=12$ h.$-$R. A.$=79^\circ$ $52'\ 30''=\alpha'$, PQE=apparent obliquity of the ecliptic $=23^\circ\ 27'\ 20''=B$, $S'OD'=SOD=90^\circ-\lambda$, $SQ=\alpha$ $=180^\circ-$sun's Long. Required DS the sun's Dec. north. Then $\sin D'O=\tan\lambda\tan\delta$, and $DO=15^\circ\ 49'$ $2''=m$, and $OQ=\alpha-m=64^\circ\ 3'\ 28''=c$, and



$$\cot\alpha=\frac{\cos(B-\gamma')\cot c}{\cos\gamma'},$$

and $a=49^\circ\ 34'\ 22''$, where $\cot\gamma'=\cot\lambda\cos c$, and $\gamma'=64^\circ\ 28'\ 47''$. Whence $\sin DS=\sin B\sin\alpha$. $\therefore\ DS=17^\circ\ 38'\ 16''$ north, and sun's Long. $180^\circ-\alpha=130^\circ\ 25'\ 38''$. The sun's Dec. and Long. found, give by the common solar Tables, the date of August 2d, eleven days before and after this event is the beginning and end of "Dog-Days."

It is well to note that the assertion of Prof. Matz, in his solution in the June number, that the hour-angles of the Dog-Star and the sun are equal, is not true. For the hour-angle of the sun must be the supplement of that of the Dog-Star, instead of equal to it. That of course makes the hour-angle of the Dog-Star from the upper meridian=that of the sun from the lower meridian, as it should have been stated.

NOTE.—On page 134, Prof. Matz asserts that when the moon is $\frac{2}{3}$ through her last quarter she will have a reversed crescent of the same size as when $\frac{2}{3}$ through her *first* quarter. This is not true. The Moon can be only $\frac{1}{3}$ through her last quarter, when such a crescent is seen.—S. H. W.

PROBLEMS.

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31. **Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.**

In order that a vertical cylindric stalk may be severed by a blow of minimum force, the stalk must be struck at what inclination by a sharp wedge-shaped blade?

32. **Proposed by S. H. WRIGHT, M. D., M. A., Ph. D., Penn Yan, New York.**

Intermittent reflections of flashes of light on a clear sky after dark, indicated a storm was progressing *below* the horizon. Refraction of $34'$ on the horizon, brought the upper edge of the storm-cloud up to the horizon, and was just visible. How far off was the storm if the cloud was one mile above the earth?

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

The American Mathematical Monthly in Spain.

The leading article of the Spanish Journal "El Progreso Mathematico" for March 1895 speaks of "la rica literatura que forman en conjunto las abras escritas en estos ultimos anos acerca de los diversos sistemas geometricos de los Sres. Flye Ste. Marie, Frischauf, Erdmann, Lipschitz, Scheffler, Killing, Battaglini, Cayley, Klein, Poincare, Bruce Halsted(* * *), Vassilief;" and the note is "(* * *) Este ilustrado profesor de la Universidad de Texas publica actualmente una serie de articulos sobre la Geometria non-Euclidea en the *American Mathematical Monthly*."

The trisection problem is considered in Petersen's Methods and Theories (London 1879) p. 101. Also in the recently published Vortrage uber ansgewahlte Fragen der Elementargeomentrie, von. F. Klein, ausgearbeitet von F. Torgert (Leipzig 1895), p. 11.

DAVID E. SMITH.

I reply to W. E. Heals inquiry on page 171 of AMERACAN MATHEMATICAL MONTHLY as to the impossibility of trisecting an angle with rule and compass as follows:

We can draw only circles and straight lines with a compass and rule and can therefore express by that means only the roots of quadratics. By means of the compass we can find the square root. But the equation $\sin 3A = 3\sin A - 4\sin^3 A$ is a cubic and we have no method of finding the cube root by means of a compass.

OTTO CLAYTON.

NOTES ON THE DEMONSTRATION OF EUCLID'S ELEVENTH AXIOM.

NOTE 1.—To complete my demonstration in the May No. GF (in *third*) should be ~~solved~~ equal to the given *constant* DA. If GF be supposed greater than DA, then GD and FA would converge toward DA, and when folded over on axis DA, the corresponding lines DM and AN would diverge, making GDM and FAN broken lines, whereas, by hypothesis, they are straight lines. Similarly, if GF be supposed less than DA.

NOTE 2.—Since mathematicians differ about the definition of parallels, might not the term *parallel* be dispensed with and *equidistant* be substituted?

WARREN HOLDEN,
Girard College, Philadelphia.

ANSWER TO "READER'S" QUERY.

Some one signing himself "Reader" in the February issue, 1895, makes inquiry as to the meaning of the word "pseudo spherical". He will find an answer to that part of his question in my article in the May No. of the MONTHLY. *Numerous* points of "difference" between the Euclidian and non-Euclidian Geometry are also indicated.

"Reader" will find the term "Hyper-space" in the article on "Measurment" by R. S. Ball in the Encyclopaedia Brittanica, Ninth Edition. I do not find the expression "Ideal Space" in either of my articles in the November issue of last year. Kant maintained that space is purely subjective. Fichte reasoned that if space is subjective, the material bodies contained in it must also be subjective. This is Idealism. The Kantian view of space may appropriately be called Idealistic inasmuch as it is avowedly purely subjective.

JOHN N. LYLE.

EDITORIALS.

THE MONTHLY will be published at Kidder until January after which it will probably be published at Springfield.

DR. F. P. MATZ has accepted the Professorship of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania. This is one of the most flourishing Colleges in Pennsylvania; and Dr. Matz is the recipient of a handsome salary, as well as the holder of a very desirable position.

DR. ALEXANDER MACFARLANE, a contributor to the MONTHLY, and a distinguished Physicist and Electrician, formerly Professor of Physics in the University of Texas, has accepted the chair of Lecturer in Electrical Engineering in Lehigh University. Dr. Macfarlane's reputation as an investigator in Mathematical Physics is international and the University to which he has been called is to be congratulated for selecting such an able man.

DR. E. S. LOOMIS, for the last ten years professor of Mathematics and principal of the Normal Department in Baldwin University, has been elected Professor of mathematics in the West High School of Cleveland. This appointment, coming as it does, at this time, from our neighboring city of Cleveland, cannot be considered otherwise than as a vindication of Dr. Loomis' eminent ability as a teacher of mathematics, and also shows that notwithstanding his doubts that Jonah "swallowed the whale," the Cleveland Educational authorities believe him still capable of teaching that science.—*Berea Advertiser.*

We congratulate the Doctor upon his good fortune in being called to a position in which his superior ability will be fully appreciated.

THE National Normal University, Lebanon, Ohio, of which Dr. Alfred Holbrook is the President, recently conferred (with the highest distinction) the Degree of *Doctor of Science* (Sc. D.), upon our valued contributor, F. P. Matz. The subjects of the theses submitted by him, were: (1) *The Conditions*

Necessary to the Existence of Life and Mind; (2) *Neo-Vitalism considered from a Monistic Standpoint;* (3) *New Methods for the Derivation of the Formulae for the Perturbations of Comets.*

DR. G. A. MILLER, Professor of Mathematics in the University of Michigan, has gone to Leipzig, Germany, to pursue his study in Mathematics. Dr. Miller is contributing, to the MONTHLY, a series of articles on *Substitution Groups* for which we are very thankful. This difficult subject is receiving a great deal of attention from eminent Mathematicians. Dr. Miller is now presenting the subject to the readers of the MONTHLY in a way that will be greatly appreciated.

DRS. ZERR and Matz both sent *strong* replies to Counselor Dolman's Comments on Problems 14 and 15, Average and Probability, but we thought best not to use any more space in the discussion. The published solution of problem 14 is unquestionably correct.

PROFS. J. F. W. SCHEFFER and A. H. Bell should have received credit for solutions of problem 49, Arithmetic Department.

WE HAVE received several calls for copies of the following books: *Salmon's Treatise on the Higher Plane Curves* (Latest Edition); *Salmon's Treatise on the Analytic Geometry of Three Dimensions* (4th Edn. 1882); *Salmon's Modern Higher Algebra; B. Price's Treatise on the Infinitesimal Calculus* (4 vol.). If any of our readers have copies of any of the above named books and wish to dispose of the same they should write to us at Springfield, Mo. Or, if any of our readers have other books and wish to dispose of them, we shall be pleased to publish the name of such books and the price of same, in the next issue of the MONTHLY. We can thus be of service to those wishing to obtain books and those wishing to dispose of them.

ALL communications and subscriptions should be sent to B. F Finkel, Springfield, Mo. Persons failing to receive their copies should write to the Publishers, Kidder, Mo.

BOOKS AND PERIODICALS.

Algebra for Beginners. By H. S. Hall and S. R. Knight. Revised and Adopted to American Schools by Frank L. Sevenoak, A. M., M. D., Professor of Mathematics and Assistant Principal in Stevens School, Academic Department of Stevens Institute of Technology.

Small 8vo. cloth, VIII+188 pp. Price, 60 cents. New York: Macmillan & Co.

This neat little work treats of all subjects to and including quadratic equations usually presented in an elementary algebra. The aim of the reviser has been to thoroughly adopt the book to the wants of all those who do not require a knowledge of Algebra beyond Quadratic Equations. In this, he has not failed. Each subject contains a long list of well selected examples, a very commendable feature in any Mathematical text-book. B. F. F

Bellum Helveticum for Beginners in Latin. By Cornelius Marshal Lowe, Ph. D., Heidelberg University, and Nathaniel Butler, Jr., M. A., University of Chicago. 8vo. cloth, 312 pp. Chicago: Albert, Scott & Co.

This book is an introduction to the reading of Latin Authors, based on the inductive method illustrating the forms and construction of classical Latin prose. The book is written to satisfy the demand of many teachers that Latin be made at the beginning, a living subject for the student. We consider the book a most excellent one and do not hesitate to recommend it to the favorable consideration of teachers of Latin. B. F. F.

The Essentials of Arithmetic, Oral and Written. By Gordon A. Southworth, Superintendent of Schools, Somerville, Massachusetts. 8vo. cloth, 186 pp. Boston, New York and Chicago: Leach, Shewell & Sandborne.

This book is designed for use in the third, fourth, fifth, and sixth-year grades of Public Schools. Oral and written exercises are happily blended throughout the entire work. The answers to the exercises are appended at the close of the book. B. F. F.

An Elementry Algebra, Theoretical and Practical. By J. W. Nicholson, A. M., President and Professor of Mathematics in the Louisana State University and Agricultural and Mechanical College. 8vo. Half Leather back, 284 pp. New York and New Orleans: University Publishing Co.

This is a splendid elementary algebra in which the author has treated a number of subjects in an original manner. The book contains about 2500 problems. B. F. F.

Elements of Geometry, after Legendre with a Selection of Geometrical Exercises, and Hints for the Solution of the same. By Charles S. Venable LL. D., Professor of Mathematics in the University of Virginia. 8vo. cloth, 413 pp. New York and New Orleans: University Publishing Co.

This is a complete translation and adaptation of the latest edition of the standard work of Legendre. A number of changes have been made. These consist mainly in the discussion of parallels; in the treatment of tangencies; in the addition of some theorems and the omission of a few; the substitution of the method of limits for the *reducto ad absurdum* in the treatment of the measure of the circle and of the "three round bodies;" etc. All these changes are for the best. Much might be said in commendation of this excellent work. B. F. F.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Albert Shaw. Price, $2.50 per year. Single Number, 25 cents. The Review of Reviews Co., New York City.

The leading features of the August *Review of Reviews* are: "Theodore Roosevelt," a character sketch by Julian Ralph; "The Clearing of Mulberry Bend," the story of the rise and fall of a New York slum, by Jacob A. Riis; "The Third Salisbury Cabinet," by W. T. Stead, and "The Record of the Rosebery Administration,"—all four articles well illustrated. The *Review of Reviews* is an illustrated summary of the world's progress. B. F. F.

The Cosmopolitan: An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year. Single Number, 10 cents.

ERRATUM.

In Dr. Martin's solution of Diophantine Problem 15, p. 439, Vol. I.

for "$(4m+3)^2=16m^2+24m+1=8(2m^2+3m)+1$"

read $4(m+3)^2=16m^2+24m+9=8(2m^2+3m+1)+1$.

DeVolson Wood.

THE AMERICAN MATHEMATICAL MONTHLY.

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BIOGRAPHY.

PROFESSOR DE VOLSON WOOD.

BY F. P. MATZ.

DE VOLSON WOOD was born near Smyrna, New York, in 1832. To 1852 he enjoyed the educational advantages of the common school, of six weeks in a private academy, and of one term in Cazenovia Seminary. He began teaching public school, in 1849; and in his native town, Smyrna, he taught three terms. In 1853, he was graduated by the Albany (State) Normal School; and during 1853 and '54, he was the Principal of Schools, Napanoch, Ulster county. He was Assistant Professor of Mathematics in the Albany Normal School, 1854-1855; Assistant Teacher and Student, Rensselaer Polytechnic Institute, Troy, 1855-1857; from which Institution he was graduated with the Degree of Civil Engineer (*C. E.*), in 1857; was *honored* by Hamilton College, in 1859, with the Degree *A. M.* He was a Professor in the University of Michigan, from 1857 to 1872; and from this University, he received the Degree of *M. Sc.*, in 1859. He served as Professor of Mathematics and Mechanics in Stevens Institute of Technology, Hoboken, New Jersey, from 1872 to 1885; and in the capacity of Professor of Mechanical Engineering, he is serving this Institute, since 1885.

Professor Wood was a member of the American Society of Civil Engineers, from 1871 to 1885. He has been a member of the American Association for the Advancement of Science, since 1879; and he was the Vice President of this Association, in 1885. Professor Wood is a member of the American Mathematical Society, and an *honorary* member of the American Society of Architects. He has been a member of the American Society of Mechanical Engineers, since 1886; was the *first* President of the Society for the Promotion of Engineering Education; and was the Engineer of the Ore-Dock, Marquette, Michigan, in 1864. He is the *inventor* of "Wood's Steam Rock-Drill," 1866 and later; and he is, also, the inventor of other machinery.

Among the articles contributed by Professor Wood, to various magazines, books, etc., may be mentioned: *Alligation*, to the "New York Teacher" —and highly commended in "Brooks's History of Arithmetic;" *Foundations*, in "Johnson's Cyclopedia;" *Mechanics*, in Appleton's Cyclopedia of Mechanics;" *Luminiferous Aether*, in the "London Philosophical Magazine"—and in Van Nostrand's Science Series, No. 85; and *Radiant Heat not an Exception to the Second Law of Thermodynamics*, in the "American Engineer."

Professor Wood has contributed to the "AMERICAN MATHEMATICAL MONTHLY," to the "Michigan Journal of Education," to the "Journal of the Franklin Institute," to the "Railroad Gazette," to the "Mining and Engineering Journal," to the "National Educator," to the "Mathematical Visitor," to the "Analyist," to "Van Nostrand's Engineering Magazine," to the "Educational Notes and Queries," to the "American Engineer," to "Science," to the "Annals of Mathematics," to the "New England Journal of Education," to the "Mathematical Magazine," to the "Engineer," to the "Barnes Educational Monthly," to the "Mathematical Messenger," etc., etc.

Professor Wood is the *author* of the following books: *Trusses, Bridges and Roofs*, published in 1872; *Wood's Edition of Mahan's Civil Engineering*, published in 1873; *Treatise on the Resistance of Materials*, published in 1875; *The Elements of Analytical Mechanics*, published in 1876; *Wood's Edition of Magnus' Lessons in Elementary Mechanics*, published in 1878; *Co-ordinate Geometry and Quaternions*, published in 1879; *Key and Supplement to the Elements of Mechanics*, and *Key and Supplement to the Mechanics of Fluids*, both published in 1884; *Trigonometry*, published in 1885; *Thermodynamics*, published in 1887 and enlarged in 1889; and *Turbines*, published in 1895.

Professor Wood was born, and raised, on a farm. In fact, until he went to the Albany Normal School, in 1852, the farm was his home. He began teaching at the age of seventeen; *paid by teaching*, the expenses of his education, and has been teaching every year since. The only position Professor Wood ever *sought*, was the first one he ever held; and the income during the three months was thirty dollars + the *privilege* of doing all the work from Principal to Janitor + the *obligation* of boarding around.

After he was a graduate of the Rensselaer Polytechnic Institute, he "wended westward" his way—not knowing whither he was going; but while traveling through Michigan, something prompted him to visit the State University. After arriving at Ann Arbor, he called at the office of the President, Dr. H. P. Tappan, who (after *the manner* of these Dignitaries) inquired about his aims and qualifications, and then asked him to teach a few days—until they heard from a recent appointee. He began teaching, that day; and, also, remained fifteen years. During this time, Professor Wood *organized* the Department of Civil Engineering; and this Department has existed since that time. Soon after he began teaching in the Michigan State University, his funds were exhausted; and one day he declared he would write home, if he had five cents;

and just then, at the door of the University Building, he saw a *dime* lying in the sand; quickly he picked it up, and wrote home at once. While Professor Wood was teaching at Napanoch, he was granted a vacation of one week; and during this week, he attended the closing exercises at the Albany Normal School. As he entered the Principal's office, the Principal greeted him thus: "Ah! I was just writing to you and offering you the Assistant Professorship of Mathematics. Will you take it?" The offer of the Assistant Professorship of Mathematics was promptly accepted; and at the opening of the next scholastic year, Professor Wood was a member of the Faculty of the Albany Normal School.

The neighbors used to say: "The stones on Mr. Wood's farm are *covered with figures* which his son, De Volson, had used in the solutions of problems."

Possibly the greatest satisfaction to Professor Wood is the pleasure and success he has had in the class room. Men, years after graduation, have complimented him on his success. They have asked for the *secret* of this success. They have asked him to tell how he inspired with labor—and why students would, in *many* cases, put twice the labor on *his* subjects rather than on the subjects of others—and why he did not *scold* his students—and why he was universally respected by his students, etc.

Brush (of Electric fame), Cleveland, Ohio, says: "Professor De Volson Wood got more genuine study out of me than any other teacher I ever was under."

The civil, mechanical, and electrical, *engineers*, *architects*, railroad *managers* and *presidents*, college *professors* and *presidents*, etc., who formerly were Professor Wood's students and who now are scattered over the whole world, would, if simultaneously "rounded up," form the most intelligent army that ever moved on the face of this mundane sphere.

Some years ago, Professor Wood went to New Mexico—he, also, visited Gunnison, Colorado; and during his visit, he was in only one place in which he could not immediately have been identified at a bank, by one of his former pupils. Stepping off the train at Topeka, Kansas, on his return, he met a former student who had been a passenger on the same train. To him Professor Wood expressed the desire of having cashed a fifty-dollar check. The former student quickly stepped to the ticket-agent, requested him to cash the check, and Professor Wood promptly received the desired fifty dollars.

The books written by Professor Wood have proved of great assistance to science, although no *radical* reforms or changes in them are attempted. Professor Wood has unceasingly sought to make the books written by himself—*his own*; and to that extent, he made them *original*. His books have found desirable places in foreign lands. He is now engaged in *enlarging* his work on Turbines; and he hopes to make it not only rigidly theoretical, but also as practical as the solution of so difficult a problem can be made by a finite mind *working* with its own products—*under the guidance of common sense and reason.*

Professor Wood was married to Miss Cordera E. Crawe, Earlville, New York, in 1859. She died in 1866; and two years after her death, he married

Miss Fannie Hartson, Mexico, New York. By the first marriage, one son was born to them, who died in 1889. The second marriage was blessed with six children, five of whom are living. Professor Wood is a member of the Methodist Church—and, also, a member of the Official Board of the Methodist Church.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton), Ph. D., (Johns Hopkins), Member of the London Mathematical Society, and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the July-August Number.]

SCHOLION I. *In which is weighed the attempt of Proclus.* After the theorems so far demonstrated by me, independently of the Euclidean postulate, toward an exact demonstration of which they should all conspire; in my judgment it is well if I diligently weigh the labors of certain well-known geometers in the same endeavor.

I begin from Proclus, of whom Clavius in the Elements after P. XXVIII, Book I, gives the following assumption:

If from one point two straight lines making an angle are produced infinitely, their distance will exceed every finite magnitude.

But Proclus demonstrates indeed (as Clavius there well remarks) that two straights (fig. 20) as suppose AH, AD going out from the same point A toward the same parts, always diverge the more from each other, the greater the distance from the point A, but not also that this distance increases beyond every finite limit that may be designated, as was requisite for his purpose.

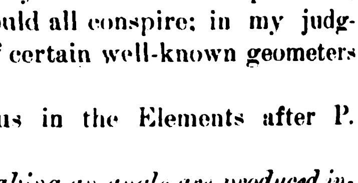

FIG. 20.

In which place the aforesaid Clavius cites the example of the Conchoid of Nikomedes, which going out from the same point A as the straight AH toward the same parts, so recedes always more from it, that nevertheless only at an infinite production is their distance equal to a certain finite sect AB standing perpendicular to AH and BC produced to infinity toward the same parts.

Why may not the same be said of the two supposed straight lines AH, AD, unless a special reason constrains to the contrary?

Nor here can Clavius be blamed that he opposes to Proklos this property of the Conchoid, which cannot be demonstrated except with the aid of many theorems resting upon the here controverted postulate.

For I say from this itself the force of the Clavian rebuttal is confirmed; for it is certain from this postulate being assumed that truly it follows manifestly, that two lines protracted to infinity, one straight, and the other curved, can recede one from the other ever more within a certain finite determinate limit; whence at any rate may arise a suspicion lest the same may be able to happen for two straight lines, unless otherwise demonstrated.

But not therefore, after I in the corollary to the preceding proposition I have made manifest the absolute truth of the aforesaid assumption, is it possible immediately to go over to the assertion of the Euclidean postulate. For previously must also be demonstrated, that those two straights AH, BC, which with the transversal AB make two angles toward the same parts equal to two right angles, as for example each a right angle, do not also, protracted toward these parts to infinity, always separate more from one another beyond all finite assignable distance. For if one chooses to presume the affirmative, which is indeed entirely true in the hypothesis of acute angle; it certainly will not be a legitimate consequence, that the straight AD in any way cutting the angle HAB, hence of course making at the same time two internal angles DAB, CBA toward the same parts less than two right angles; that, I say, this straight AD, produced to infinity must at length meet with BC produced; even if it were at another time demonstrated, that the distance of the two AH, AD produced to infinity ever greater goes out beyond all finite limit that may be assigned.

But that the aforesaid Clavius should have judged the truth of this assumption sufficient for demonstrating the postulate here in question; that ought to becondoned because of the opinion preconceived by Clavius about equidistant straight lines, which we may discuss more conveniently in a subsequent Scholion.

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

(Continued from the July-August Number.)

Construction of Intransitive Groups.

Suppose that we have an intransitive group (G) involving the six letters a, b, c, d, e, f and that in this group a is replaced by b, c, and d, but not by e or f. This group must have at least one substitution (s_1) in which a is replaced by b, one (s_2) in which a is replaced by c and one (s_3) in which a is replaced by d. In some power of s_1 (which, from the definition of a group, must also be in G) b is replaced by a.* Let this substitution be denoted by s'_1, and consider the following substitution of G:

$$s'_1, \quad s'_1 \; s_2, \quad s'_1 \; s_3.$$

In the first of these b is replaced by a, in the second by c and in the third by d. Hence we see that the hypothesis that a is replaced by each of the

*Suppose s_1 were one of the following substitutions: ab, abc, $abcd$, $ab.cd$; then b would be replaced by a in the first power of the first and last substitutions, in the second power of the second substitution and the third power of the third.

other letters in the first set of four requires that the other letters of this set or system have the same property, for what we proved in regard to b can be proved, similarly, in regard to c and d.

It is further evident that there can be no substitution in G which replaces one of these four letters by e or f; for suppose that the substitution represented by s_α has this property, replacing, for instance, b by e, then would $s_1 t$ replace a by e, which is contrary to hypothesis. Hence e and f can only replace each other in the substitutions of G. Since this proof could clearly be used with respect to any number of letters we have arrived at the following important theorem:

Theorem 1. In every intransitive group the letters may be divided into such systems that the substitutions of the group will connect the letters of the systems transitively and no substitution will replace a letter of one system by one of another system.†

Since each of these systems must form a transitive group it follows that *every intransitive group can be formed by combining transitive groups, such that the sum of the letters in these groups is equal to the number of letters in the intransitive group.*

The main problem before us is, therefore, the development of such methods as are most helpful in combining the transitive groups. The two intransitive groups in the list of the groups of four letters are instances of the simplest methods of combination. One of these

$$1,\ ac,\ bd,\ ac.bd$$

is obtained by multiplying every substitution of one group (ac)* by every substitution of the other (bd). It is evident that by this method we can always construct an intransitive group from two or more groups each involving different letters, *e. g.* the intransitive group of five letters which may be obtained in this way from (abc) and (de) is

$$1,\ abc,\ acb,\ de,\ abc.de,\ acb.de$$

The other intransitive group in the given list

$$1,\ ab.cd$$

is obtained by the process called simple *isomorphism* or 1, 1 correspondence. The process consists in associating substitutions of the component transitive groups which have the same properties with respect to the groups. The given intransitive group is obtained from the groups

$$\begin{matrix} 1 & 1 \\ ab & cd \end{matrix}$$

It is at once evident that 1 may be associated with 1 and ab with cd, and we thus obtain the required group

$$\begin{matrix} 1 \\ ab.cd \end{matrix}$$

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†cf. Jordan: Traite des Substitutions, Art. 40: also Netto's Theory of Substitutions (Cole's edition), page 70.

*The parenthesis is used to indicate the group generated by the substitution enclosed: thus, $(ac) = 1, ac$; $(abc) = 1, abc, acb$; etc. In case the group consists of more than two substitutions the abbreviation for cyclical is commonly written after the parenthesis: thus, $(abc) = (abc)\ cyc.$

Similarly we may obtain an intransitive group of six letters from (abc) and (def), viz.

$$1,\ abc.def,\ acb.dfe$$

Instead of a 1, 1 correspondence we may have an a, b correspondence, a and b representing any positive integers. In this way we obtain the intransitive group

$$1,\ abc,\ acb,\ ab.de,\ ac.de,\ bc.de$$

from the two groups

$$1,\ abc,\ acb,\ ab,\ ac,\ bc$$

and $1,\ de$

by a 3, 1 correspondence.

We have now given the main methods employed in constructing intransitive groups. We proceed to find all the

Intransitive Groups of Five Letters.

All these groups are composed of a transitive group of three and another of two letters: for the only way of dividing five so as to get two or more letters in each system is to divide it into the parts three and two. The component groups are therefore,

$$\left.\begin{array}{l}1,\ abc,\ acb\\ 1,\ abc,\ acb,\ ab,\ ac,\ bc\end{array}\right|\ 1,\ de$$

It is evident that (de) can be combined with (abc) only by multiplying the two groups together. We thus obtain the intransitive group given above as an illustrative example.

By combining (de) with the second group on the left we obtain, in a similar way,

$$\begin{array}{llllll}1 & abc & abc.de & ab.de & de\\ & acb & acb.de & ac.de & ab\\ & & & bc.de & ac\\ & & & & bc\end{array}$$

In this case the combination may be effected in one more way. Since the first half of the substitutions in the second group on the left form a subgroup we may let 1 of (de) correspond to these and de to the remaining substitutions. We thus obtain the intransitive group of five letters which was given above as an illustrative example of an a, b correspondence.

From this we see that there are only three intransitive groups in five letters. In Professor Cayley's list these groups are denoted by

$$(abc)cyc.(de),\quad (abc)all(de),\quad \{\,(abc)all(de)\,\}\text{pos}.$$

There are twenty-one intransitive groups involving six letters. The component groups may involve any of the following systems of letters:

2, 2, 2,

4, 2

3, 3

We expect to apply the given methods to the construction of all these groups. This, it is believed, will give sufficient exercise in the construction of this class of groups and we shall then proceed to the construction of the *transi-*

tive groups.

The reader who desires a thorough working knowledge of this subject could very profitably work over this field and compare his methods and results with those that we shall give.

THE GOLDEN SECTION.

By EMMA C. ACKERMANN, Instructor in Mathematics, Michigan State Normal School.

In the number for November 1892, of *Lehrproben und Lehrgange aus der Praxis der Gymnasien und Realschulen*, there appeared an article by Prof. Dr. O. Willman, entitled Der goldene Schnitt abs ein Thema des mathematischen Unterrichts. The article is interesting not alone to students of geometry, but to all who are at all concerned with the question of concentration, a question which is at present creating for itself an active interest among all educators. The article is a resume of a work on the golden section by F. C. Pfeifer, Der goldene Schnitt und dessen Erscheinungsformen in Mathematik, Natur, und Kunst, Augsburg, Huttler, 1885. The statements here presented are intended as a summary of the article.

It is very necessary that the connection between instruction in mathematics and in the remaining courses of study should be carefully considered because the subject of mathematics is an abstract one and according to its nature tends towards isolation.

To assist in bringing about this connection, there should be prepared mathematical problems and exercises which will show the application of mathematics to technics and to observations in nature on the one hand, and on the other furnish an insight into the history of mathematics, by means of which historical and classical instruction can be connected with the mathematical. A subject which meets these demands and is at the same time well adapted for purely mathematical instruction is the theme of the golden section, a theme which does not appear in a complete form in our modern text-books.

The simplest division of any magnitude, involving the fewest conditions is the division into two equal parts. Calling a line so divided, S, the parts p, we have $S=2p$, $p=\frac{S}{2}$, $\frac{S}{p}=2$, $\frac{p}{S}=\frac{1}{2}$, $\frac{p}{p}=1$. Contrasted to one case of division into two equal parts stands an infinite number of divisions into two unequal parts; and the ratio of the smaller (m) to the larger (M), $\frac{M}{m}$, or the ratio of one of the parts to the whole, $\frac{m}{S}$ or $\frac{m}{m+M}$ and $\frac{M}{S}$ or $\frac{M}{m+M}$ can be expressed by many different numbers. In one case only is there no need of figures to determine the ratio of the parts to the whole; and that is when

$\frac{m}{M}=\frac{M}{m+M}$. Such a division constitutes the golden section.

From the proportion $m:M=M:m+M$, we have $mM+m^2=M^2$ or $(M+m)(M-m)=mM$; or in the golden section, the sum of the parts multiplied by their difference equals their product. It also follows that the greater part is the geometric mean between the smaller part and the whole. Let $S=m+M$; then $S-m=\sqrt{Sm}$; or the difference between the whole and the smaller part is the geometric mean of those two parts. Also both parts form with the whole a continued proportion, $m:M=M:m+M$ or $m:M:m+M$, distinguished from all other proportions by the fact that the third quantity is at the same time the sum of the other two. From $\frac{M}{m}=\frac{M+m}{M}$, we have $\frac{M}{m}=1+\frac{m}{M}$ or $\frac{m}{M}=\frac{M}{m}-1$; that is, the quotient of the two parts is greater or less than its reciprocal by unity. Call the ratio $\frac{M}{m}$, e, then we have the equation $e=1+\frac{1}{e}$, solving

$e=\frac{\sqrt{5}+1}{2}=1.61803+$.

Of the three elements of the golden section, each two can be expressed by the third. The simplest is that of m and S by means of M; $m=\frac{\sqrt{5}-1}{2}M$; $S=\frac{\sqrt{5}+1}{2}M$. Also $m=\frac{3-\sqrt{5}}{2}S$ and $M=\frac{\sqrt{5}-1}{2}S$, that is, the major part becomes the minor, when the whole is considered the major. If m is the base, $M=\frac{\sqrt{5}+1}{2}m$, $S=\frac{\sqrt{5}+3}{2}m$.

For the construction and consideration of the golden section, the number 5, which appears in the value of e is very suggestive. $5=1+4$, and therefore can be expressed by the Pythagorean theorem; 5 is the area of the square on the hypotenuse, if $1+4$ are the squares on the other two sides. The sides themselves are 1 and 2, and the hypotenuse, $\sqrt{5}$. Half of $\sqrt{5}$ increased by $\frac{1}{2}$ of unity will then express the value of e, and diminished by $\frac{1}{2}$ of unity, its reciprocal. A line AB is divided into medial section therefore, if in the right triangle ABC, with right angle ABC, we make $BC=\frac{1}{2}AB$, draw AC, lay off $CD=BC$, and then lay off the remainder AD or AB as AE; then AB is divided into medial section.

The number 5 may be used in another way to illustrate the ratio of m to M. An isosceles triangle with angles $\frac{\pi}{5}$, $\frac{2\pi}{5}$, $\frac{2\pi}{5}$ is constructed. By bisecting one of the equal angles, we have a triangle similar to the first. Let the triangles be ABC and ABD respectively, angle B being $\frac{\pi}{5}$; then CB is divided in medial section, from principles of similar triangles. The triangle ABC is middle part of a pentagon which is completed by placing two triangles congruent to triangle ABD on AB and BC. AD produced will then pass through an angle of the pentagon and CD becomes the smaller part and BD the larger part of a diagonal. Therefore in a regular polygon of five sides the smaller part of a diagonal cut off by a second diagonal forms with the side of the pentagon and the diagonal, the proportion of the golden section. The triangle is also an element of the regular decagon and will produce it if repeated ten times.

In the division of magnitudes into two equal parts, the whole may be considered as one of the parts repeated; so in the golden section, each one of the parts may be considered as the starting-point and the next as a repetition of it augmented or diminished. If we proceed from the minor part, the major is a repetition of the minor increased, and the sum of the two bears the same relation to the major, the ratio in each case being e. So if we proceed from the whole to the major, and from that to the minor. With this view of the case, there is no necessity for stopping with three elements, since this augmenting or diminishing repetition can evidently be carried on indefinitely. In this way the geometric proportion of the golden section becomes a geometric progression which from analogy is called the golden progression, the ratio being e or $\frac{1}{e}$. The golden progression differs from the other geometric series in this that each of its members is also the sum of the two preceding.

If its first term is a, then this progression has the form a, ae, ae^2, ae^3, $ae^4 \ldots . ae^n$. or a, ae, $a+ae$, $a+2ae$, $2a+3ae$, $3a+5ae \ldots .$

Then $ae^2=a(1+e)$; $ae^3=a+2ae$; $ae^4=a(2+3e)$, etc. If $a=1$, then since $e=\frac{\sqrt{5}+1}{2}=1.61803$, the series is: 1, 1.61803+, 2.16803+, 4.23607+, 6.185410+, 11.09017+, 17.94427+, 29.03444+, etc. If we should tentatively place e also equal to 1, that is, 0.61803 too small, then as a geometric progression simply, the series remain stationary. But using the other property of the golden progression, the series becomes 1, 1, 2, 3, 5, 8, 13, etc. The quotient of any two successive members is alternately smaller and larger than e, but as the series advances, the quotient approaches nearer to e, as at $\frac{233}{144}=1.618$. This shows that a series beginning with the smallest natural numbers and advancing according to the second condition above, forms in its continuation an approximation to the golden progression.

The golden progression can be represented graphically thus: A pentagon is drawn whose sides are F and diagonals D; the intersections of the diagonals determine a pentagon whose sides are f and whose angles are at a distance a from the vertices of the original polygon. Then f, a, F, D form a golden series. By making the original polygon the enclosed polygon of a larger pentagon, whose sides and diagonals are F' and D' respectively, we can continue the progression as an ascending series; or, by drawing polygons within the pentagon, as a decending series.

Another interesting figure might be given here. The line AB is divided in medial section at C, AC being the major part; perpendiculars BD,

CF, of length AC are drawn at B and C, and are divided at E and G respectively into medial section, BE and GC being the major parts. Then AB, AC $(=BD=CF)$, $BC(=BE=CG)$ and $DE(=FG)$ form a golden series. Connect F and A, G and A, E and A. Let AC be taken as a radius or unity; AC then represents the side of a hexagon; AF the side of a square; AG, of a regular pentagon, EB or CG, the side of a decagon; and AE, the side of a regular triangle, all of which is evident from the right angled triangle.

Another figure may be obtained by making m, M, $m+M$ the radii of concentric circles. Their areas are then πm^2, πM^2, $\pi(m+M)^2$, the area of the inner ring, r, is $\pi(M^2-m^2)$ and of the outer ring, R, $\pi[(M+m)^2-M^2]$. If $m=1$, then $M=e$, $m+M=e^2$, $(M+m)^2=e^4$, $M^2-m^2=e^2-1$, and $(M+m)^2-M^2=e^4-e^2=e^2(e^2-1)=e^3$. Then the following series arises:

$\pi=$area of inner circle, f;
$e\pi=$ " " " ring, r;
$e^2\pi=$ " " middle circle, F;
$e^3\pi=$ " " outer ring, R;
$e^4\pi=$ " " " circle, F'.

This series can be extended both as increasing and decreasing; the members with even exponents as e^6, e^8, e^{-2}, e^{-4} correspond to circles; those with odd exponents to rings.

Countless illustrations of the proportions of the golden section are found in nature and the works of man. The golden section follows closely upon bisection (the basis of symmetry) everywhere, and the forms which are based upon the proportions of the golden section though not so evident are more widely distributed than would appear at first thought. Whenever, in the products of art or manufacture, there is no equal division, (symmetry), the artist or workman unconsciously employs the proportions of the golden section. Irregular inequality and capricious division is disagreeable to both eye and hand; and the porportion of the golden section seem to be the only acceptable ones. Accordingly, the form of writing-paper, books, a page of the MATHEMATICAL MONTHLY, furniture, especially tables and chairs, doors, windows, dimensions of pictures, foundations and often the facades of buildings, all reveal these proportions.

This is true of not only modern art and technics, but also of the ancient. We find the same proportions in the pyramids of Cheops, in the temples at Karnak and at Ombos, in the Grecian temples, and many cathedrals.

In verses of poetry and in music, the same relation is found, and most abundantly in nature. In leaves, plants, lower animals, and man, these proportions have been verified.

The subject of the golden section is not discussed by Euclid; he had a knowledge of it and mentions it in his works, though not under this name. The name, though it has an ancient ring, is not found in ancient literature. Aristotle does not mention the subject, but it is claimed that in his philosophical reasoning, there rules the principle of the golden section; *i. e.*: the relation of the whole to the part and the parts to each other. His ideas were not carried out by the ancient philosophers but they were the source of much of the speculation in mediaeval times, when mathematical and philosophical thought were closely allied. One writer, John Campanus of Novava, thought that the princiciple of the golden section descended from the gods. Kepler compared it to a precious stone, and called it *proportio divina*, but not *proportio* or *sectio aurea*. The latter name has originated since his time.

THE RECTIFICATION OF THE CASSINIAN OVAL BY MEANS OF ELLIPTIC FUNCTIONS.

By F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

(Continued from the July-August Number.)

The central-polar equation of the Cassinian Oval may be written

$$r^4-(2c^2\cos 2\theta)r^2=m^4-c^4\ \ldots\ (1).$$

$$\therefore \cos 2\theta=\frac{r^4-(m^4-c^4)}{2c^2r^2}, \text{ and } \sin 2\theta=\sqrt{\left(\frac{4c^4r^4-[r^4-(m^4-c^4)]}{4c^4r^4}\right)}.$$

$$\therefore \mathsf{P}=8m^2\int_b^a\frac{r^2dr}{\sqrt{\{4c^4r^4-[r^4-(m^4-c^4)]^2\}}}$$

$$=8m^2\int_b^a\frac{r^2dr}{\sqrt{\{[(m^2+c^2)^2-r^4]\times[r^4-(m^2-c^2)^2]\}}}\ \ldots\ (2).$$

Reducing (2) under the supposition that

$$r^4=(m^2+c^2)^2\cos^2\phi+(m^2-c^2)^2\sin^2\phi,$$

$$\mathsf{P}=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{r}=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{[(m^2+c^2)^2\cos^2\phi+(m^2-c^2)^2\sin^2\phi]^{\frac{1}{4}}}$$

$$=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{[(m^2+c^2)^2-4m^2c^2\sin^2\phi]^{\frac{1}{4}}}\ \ldots\ (j),$$

$$=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{[(m^4+c^4)+2m^2c^2(1-2\sin^2\phi)]^{\frac{1}{4}}}$$

$$=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{[(m^4+c^4)+2m^2c^2\cos 2\phi]^{\frac{1}{4}}}\ \ldots\ (3).$$

Let $2\phi=\psi$, and make $2m^2c^2/(m^4+c^4)=C$; then, after obvious transformations, (3) gives

$$\mathsf{P}=\frac{2m^2}{[m^4+c^4]^{\frac{1}{2}}}\int_0^{\pi}\frac{d\psi}{[1+C\cos\psi]^{\frac{1}{2}}}\ldots.(4).$$

After expanding (4) into a series of not less than two dozen terms, and observing that the negative terms of the series will *vanish* on taking the integral limits, we obtain a series expressing the perimeter of the Cassinian Oval. Since $m^2=5$ and $c^2=4$; that is, since the semi-axes of the Cassinian Oval in consideration are 3 and 1 linear units, we have $C=\frac{4}{4}\frac{0}{1}$. After a rather laborious calculation, we find $\mathsf{P}=14.9831+$ linear units. On page 223 of the July-August MONTHLY, $C=\frac{8}{8}\frac{0}{1}$; and four terms of that resulting series give a perimeter ($\mathsf{P}=12.7329+$ linear units) too small by $2\frac{1}{4}$ linear units. Since the *moduli* of these functions are almost unity, the resulting series will not converge rapidly; and with this same trouble, it must be remembered, *M.* Legendre also had to contend. Possibly some of the talented readers of the MONTHLY will succeed in expanding (j), or (4), into a *rapidly-converging* series.

After performing certain rather elaborate transformations of *premises approximative in derivation*, we deduce the following two remarkable formulae for the perimeter of the Cassinian Oval:

II. Transforming the Cartesian equation of the Cassinian Oval by the formulae, $x=r\cos\theta$ and $y=r\sin\theta$, we have

$$r^2=\surd(m^4-c^4\sin^2 2\theta)+c^2\cos 2\theta\ldots.(1).$$

$$\therefore\ rdr=\frac{-c^2[\surd(m^4-c^4\sin^2 2\theta)+c^2\cos 2\theta]\sin 2\theta d\theta}{\surd(m^4-c^4\sin^2 2\theta)}\ldots.(\alpha),$$

$$\text{and}\left(\frac{dr}{d\theta}\right)^2=\frac{c^4[\surd(m^4-c^4\sin^2 2\theta)+c^2\cos 2\theta]\sin^2 2\theta}{m^4-c^4\sin^2 2\theta}\ldots.(\beta).$$

$$\therefore\ \mathsf{P}=4m^2\int_0^{\frac{1}{2}\pi}\sqrt{\left(\frac{\surd(m^4-c^4\sin^2 2\theta)+c^2\cos 2\theta}{m^4-c^4\sin^2 2\theta}\right)}d\theta\ldots.(2).$$

Put $(c^2/m^2)^2=C^2$; then (2) can easily be transformed into

$$\mathsf{P}=4m\int_0^{\frac{1}{2}\pi}\sqrt{\left(\frac{\surd(1-C^2\sin^2 2\theta)+C\surd(1-\sin^2 2\theta)}{1-C^2\sin^2 2\theta}\right)}d\theta$$

$$=4m\surd(1+C)\int_0^{\frac{1}{2}\pi}\sqrt{\left[1+\left(\frac{2(1+C^2)-(2+C)}{2}\right)\sin^2 2\theta\right]}d\theta$$

$$=4m\sqrt{\left(\frac{(1+C)[2(1+C^2)-C]}{2}\right)}\int_0^{\frac{1}{2}\pi}\sqrt{\left[1-\left(1-\frac{2}{2(1+C^2)-C}\right)\sin^2\phi\right]}d\phi,$$

$$=\tfrac{3}{2}\pi m\sqrt{\left(\frac{(1+C)[2(1+C^2)-C]}{2}\right)}\left[1-\Sigma\left(\frac{1.3.5\ldots.(2n-1)}{2.4.6\ldots.2n}\right)^2\frac{M^{2n}}{2n-1}\right];$$

of which elliptic function M is the *modulus*, and n may have all consecutive integral values from *unity* to *infinity*. When $m^2=5$ and $c^2=4$, $\mathsf{P}=14.9652$.

III. From the Cartesian equation of the Cassinian Oval, we deduce

$$y^2=\sqrt{(m^4+4c^2x^2)}-(c^2+x^2)\ldots\ldots(3),$$

an equation which gives all the *real* points of the Oval in consideration.

$$\therefore \left(\frac{dy}{dx}\right)^2=\frac{[4c^4-4c^2\sqrt{(m^4+4c^2x^2)}+m^4+4c^2x^2]x^2}{(m^4+4c^2x^2)[\sqrt{(m^4+4c^2x^2)}-(c^2+x^2)]}\ldots\ldots(\gamma).$$

Representing the semi-axis major of the Cassinian Oval by $a,=\sqrt{(m^2+c^2)}$, we have

$$P=4\left(\frac{2c^2-m^2}{m^2\sqrt{(m^2-c^2)}}\right)\int_0^a\sqrt{\left(\frac{m^4(m^2-c^2)}{(2c^2-m^2)^2}+x^2\right)}dx$$

$$=2\left[\left(\frac{\sqrt{[m^4(m^2-c^2)+(m^2+c^2)(2c^2-m^2)^2]}}{m^2}\right)\sqrt{\left(\frac{m^2+c^2}{m^2-c^2}\right)}\right.$$

$$+\frac{m^2\sqrt{(m^2-c^2)}}{2c^2-m^2}$$

$$\left.\log\left(\frac{(2c^2-m^2)\sqrt{(m^2+c^2)}+\sqrt{[m^4(m^2-c^2)+(m^2+c^2)(2c^2-m^2)^2]}}{m^2\sqrt{(m^2-c^2)}}\right)\right],$$

$$=\tfrac{16}{9}\left[\frac{3\sqrt{(106)}}{5}+\tfrac{5}{3}\log\left(\frac{9+\sqrt{(106)}}{5}\right)\right]=14.9833,$$

when $m^2=5$ and $c^2=4$.

[To be continued.]

POSTULATE II. OF EUCLID'S ELEMENTS.

By Professor JOHN N. LYLE, Ph. D., Westminister College, Fulton, Missouri.

"Let it be granted that a terminated straight line may be produced to any length in a straight line."

Euclid lays down the statement just quoted as his second postulate regulative of geometrical constructions. Wherever in unbounded space any point may be located to which a straight line has been extended, Euclid assumes that the straight line may be lengthened out beyond that point.

Riemann assumes that every straight line is finite in length, and if extended will ultimately return to the starting point.

If a straight line that is produced from a given point eventually returns to the same point, Euclid's postulate 2 is false.

On the other hand, if the second postulate of Euclid is true, the Riemannian hypothesis that contradicts it must be false. This follows inevitably by the logical law of *Excluded Middle*, according to which if one of two propositions that mutually contradict each other is true, the other must be false.

According to the Euclidian view the longer a straight line is the further apart are its ends.

According to the Riemannian view a straight line may be lengthened until its ends approach and ultimately meet.

The hypothesis of Riemann and the 2nd postulate of Euclid contradict each other. Hence, both cannot be true. To accept both is to discredit logical law. To say that we do not know which is true is to confess that we are not in possession of geometrical Science.

According to the laws of logical deduction, if Euclid's postulate 2 is false, the geometrical System derived from it is not true.

On the other hand, if the assumption that contradicts Euclid's postulate 2 is false, the system logically deduced from it is not true. Sound geometrical propositions are not obtained by logical deduction from false data.

According to the Riemannian hypothesis the angle sum of a rectilineal triangle is greater than two right angles. But Lobatschewsky proves in his theorem 19 that the angle sum of a rectilineal triangle cannot be greater than two right angles. The hypotheses of Lobatschewsky and Riemann, therefore, are seen to clash with each other as well as with the axioms, postulates and theorems of Euclid's Elements.

The chords of arcs of circles are not identical with the arcs subtended by them. Hence *rectilineal* triangles should not be treated as identical with *spherical* triangles. This statement holds whatever the length of the radius of the sphere may be. The radius of every sphere has *two* ends, one at the centre and the other at the surface. But every straight line with *two* ends is *finite*. We are now face to face with Postulate III. of Euclid's Elements.

SUBSTITUTION GROUPS.

THE CONSTRUCTION OF INTRANSITIVE GROUPS CONTINUED.

Before seeking all of the possible intransitive groups of degree* six it seems well to call attention to several facts which may be employed to advantage in this work. To illustrate we shall employ a group which was given before, viz.

* The degree of a group is equal to the number of letters it involves. Thus $(abcd)$ pos is of the fourth degree.

		abc	*acb*
(*abcd*)pos.† = 1	*ab.cd*	*bdc*	*bcd*
	ac.bd	*adb*	*abd*
	ad.bc	*acd*	*adc*.

If t and s represent any two substitutions then is

$$t^{-1}st$$

(where t^{-1} represents the substitution which reverses the operation indicated by t) called the *conjugate of s with respect to t.* t and t^{-1} are said to be the *inverse* of each other, since $tt^{-1}=t^{-1}t=1$.

We proceed to find the conjugates of *ab.cd* with respect to the other substitutions of (*abcd*) pos. We obtain the following results:

ac.bd	*ab.cd*	*ac.bd* = *ab.cd*
abc	*ab.cd*	*acb* = *ac.bd*
acb	*ab.cd*	*abc* = *ad.bc*‡

It can readily be verifid that all the substitutions in one of the above columns transform *ab.cd* into the same substitution.

Definition. If all the substitutions of a group transform all the substitutions of a subgroup into substitutions of the subgroup, the subgroup is called a self conjugate subgroup of the given group.

1, *ab.cd*, *ac.bd*, *ad.bc*

constitute a self conjugate subgroup of (*abcd*) pos, while the subgroups

1, *ab.cd* and 1, *abc*, *acb*

are not self conjugate. If we exclude identity and the entire group from the subgroups, it can easily be verified that only one of the eight subgroups of the given group is self conjugate.

Since all the intransitive groups of a given degree n can be obtained by combining transitive groups such that the sum of their degrees is equal to n it follows that all the intransitive groups of degree six can be found by combining

(1) a transitive group of degree three with a transitive group of degree three,

(2) a transitive group of dregree two with a transitive group of degree four, and

(3) a transitive group of degree two with two transitive groups of degree two.*

We proceed to find the intransitive groups for each of these divisions separately. We shall thus not only find all the groups but also each group only once since it is evidently impossible for one group to belong to two of these divisions.

[To be continued.]

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†By (*abc....l*) all we mean all the substitutions that can be formed with the letters a, b, c,....., l and by (*abc....l*) pos we mean the subgroup of the preceding group which involves only its positive substitutions; i. e. all its substitutions which indicate an even number of interchanges of two letters. In place of (*abc*) pos it is customary to write (*abc*) cyc or merely (*abc*).

‡The conjugate may be obtained by replacing each letter by the letter which follows it in the substitution with respect to which it is conjugate. For let $s=a_1\ a_2\ a_3 \ldots.$ and let $b_1\ b_2\ b_3 \ldots$ be the letters in order which in t succeed the given letters of s. Then $t^{-1}st$ replaces b_1 by a_1, a_1 by a_2, and a_2 by b_2: i. e. it replaces b_1 by b_2 and similarly it replaces b_2 by b_3. If a_α is not explicitedly found in t we have to observe that $b_\alpha=a_\alpha$ in using this method.

*Every group is transitive whose degree (n) satisfies the inequality $n<4$.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

50. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.**

If A can walk to the city and ride back, he will require $m=5\frac{1}{4}$ hours: but if he walk both ways, he will require $n=7$ hours. How many hours will he require to ride both ways?

I. **Solution by A. L. FOOTE, Middleburg Connecticut.**

Taking it for granted that he can walk or ride either way with equal facility, we find that he could walk to the city in $\frac{n}{2}=3\frac{1}{2}$ hours and can ride back in $m-\frac{n}{2}=5\frac{1}{4}-3\frac{1}{2}=1\frac{3}{4}$ and he can also ride to the city in $1\frac{3}{4}$ hours, so he will take $1\frac{3}{4}\times2=3\frac{1}{2}$ hours. On any other supposition the problem is indeterminate.

II. **Solution by H. C. WILKS, Murraysville, West Virginia, and J. F. W. SCHEFFER, A. M., Hagerstown, Maryland**

A can walk up and walk back in $n=7$ hours. He can walk up and ride back in $m=5\frac{1}{4}$ hours.

$\therefore$ times of walking back and riding back differ by $n-m=1\frac{3}{4}$ hours.

Also times of walking round trip and riding round trip differ by $2(n-m)=3\frac{1}{2}$ hours. But he *walks* round trip in $n=7$ hours.

Hence he *rides* round trip in $n-2(n-m)=7-3\frac{1}{2}$ or in $2m-n=3\frac{1}{2}$ hours.

III. **Solution by Professor P. S. BERG, Larimore, North Dakota.**

To walk one way he will require $\frac{n}{2}$ hours. Hence to ride one way he will require $\left(m-\frac{n}{2}\right)$ hours, and to ride both ways he will require

$$2\left(m-\frac{n}{2}\right)=3\tfrac{1}{2} \text{ hours.}$$

IV. **Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Vice President in Texarkana College, Texarkana, Arkansas.**

$\frac{n}{2}$ = number of hours to walk one way, and

$m-\frac{n}{2}=\frac{2m-n}{2}$ = number of hours to ride one way.

$\therefore 2\left(\frac{2m-n}{2}\right)=2m-n$ = number of hours to ride both ways.

But $m=5\frac{1}{4}$, $n=7$.

$\therefore 3\frac{1}{2}$ hours = required time.

This problem was also solved by *Professor COOPER D. SCHMITT*, ——, and the *PROPOSER*.

51. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.**

A banker, in discounting a note due in $m=4$ months at $r=3\%$ per annum charges $C=C'12\frac{1}{4}$ more than the true discount. What is the face of the note discounted?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Vice President and Professor of Mathematics in the Texarkana College, Arkansas.

$$m \text{ months } +3 \text{ days grace} = \frac{10m+1}{10} \text{ months.}$$

$$\frac{(10m+1)r}{12000} = \text{bank discount on \$1 at } r\% \text{ for } \frac{10m+1}{10} \text{ months.}$$

$$\frac{1200+mr}{1200} = \text{amount of \$1 at } r\% \text{ for } m \text{ months.}$$

$$\frac{1200}{1200+mr} = \text{proceeds of \$1 at } r\% \text{ for } m \text{ months.}$$

$$1-\frac{1200}{1200+mr} = \frac{mr}{1200+mr} = \text{true discount.}$$

$$\frac{(10m+1)r}{12000} - \frac{mr}{1200+mr} = \frac{(1200+10m^2r+mr)r}{12000(1200+mr)} = \text{difference.}$$

$$\therefore \frac{12000(1200+mr)C}{(1200+10m^2r+mr)r} = \text{face of note.}$$

Substituting $m=4$, $r=3$, $C=12\frac{1}{4}$, we get face of note=\$35099.29.

II. Solution by the PROPOSER.

Represent the face of the note by X, and the number of days of grace by g; then the *bank discount* is $(30m+g)rX/36000$, and the *true discount* is $mrX/(1200+mr)$. According to the problem,

$$\left[\frac{30m+g}{36000} - \frac{m}{1200+mr}\right]rX = C \ldots (1),$$

$$\therefore X = \frac{36000(1200+mr)}{[(30m+g)mr+1200g]r} \text{ of \$}C, = \$35099.2908,$$ which is the face of the note required.

NOTE—Make $g=0$; then $X=123725.00$, which is the result of Miscellaneous Problem, No. 844, *Lock and Scott's Arithmetic for Schools*, page 281.

III. Solution by COOPER D. SCHMITT. Professor of Mathematics in University of Tennessee, Knoxville, Tennessee.

Interest for 4 mon. 3 da. at 3% on any principal is $\frac{123}{12000}$ of the principal and bank discount is same as simple interest.

In true discount, the three days of grace are not counted.

The true discount on any principal for $\frac{1}{3}$ of a year at 3 per cent. is the same as for one year at 1%, which is $\frac{1}{101}$ of the principal.

Hence, $(\frac{123}{12000}-\frac{1}{101})$ of the principal=\$12$\frac{1}{4}$, or $\frac{423}{1212000}$ of principal=\$12$\frac{1}{4}$, and the principal=$\frac{1212000}{423}$ of $12\frac{1}{4}$=\$35099.29.

This problem was solved with different results by *P. S. BERG*, *J. F. W. SCHEFFER*, *A. L. FOOTE* and——.

PROBLEMS.

54. Proposed by D. P. WAGONER, A. B., Principal of the School of Languages, Westerville, Ohio.

A man bought a farm for \$6,000 and agreed to pay for it in four equal annual installments at 6% annual interest compounded every instant. Required his annual payment. *B. F. Burleson.*

55. Proposed by J. C. CORBIN, Pine Bluff, Arkansas.

How long will it take to count a million, in the following manner: the counter is to pronounce each syllable in the names of the successive numbers at the rate of one per second?

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

46. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Find θ from $\cos\theta+\cos 3\theta+\cos 5\theta=0$ ....(1).

Solutions by JOE A. JOHNSON, Student of the Sophomore Class, University of Mississippi; E. L. SHERWOOD, A. M., Professor of Mathematics, Mississippi Normal College, Houston, Mississippi; J. B. FAUGHT, A. B., Indiana University, Bloomington, Indiana; J. C. CORBIN, Pine Bluff, Arkansas; G. I. HOPKINS, Department of Mathematics in High School, Manchester, N. H.; OTTO CLAYTON, M. B., Maxwell, Indiana; O. W. ANTHONY, M. S., Missouri Military Academy, Mexico, Missouri; and A. H. BELL, Hillsboro, Illinois.

I. Equation (1) $=\cos\theta+(4\cos^3\theta-3\cos\theta)+(16\cos^5\theta-20\cos^3\theta+5\cos\theta)=16\cos^5\theta-16\cos^3\theta+3\cos\theta=\cos\theta(16\cos^4\theta-16\cos^2\theta+3)=0$ ....(2). Whence $\cos\theta=0$. $\therefore\ \theta=\frac{\pi}{2}$. From $16\cos^4\theta-16\cos^2\theta+3=0$, we get $\cos^2\theta=\frac{3}{4}$, or $\frac{1}{4}$. $\therefore\ \cos\theta=\pm\frac{1}{2}\sqrt{3}$, $\pm\frac{1}{2}$. $\therefore\ \theta=\frac{\pi}{6}$, $\frac{5}{6}\pi$, or $\frac{\pi}{3}$ and $\frac{2}{3}\pi$, which are all the values $<\pi$. [*J. A. Johnson.*]

II. Factoring (2), $(4\cos^2\theta-3)(4\cos^2\theta-1)\cos\theta=0$. Whence $\cos\theta=\pm\frac{1}{2}\sqrt{3}$, $\pm\frac{1}{2}$, 0. $\therefore\ \theta=\frac{\pi}{6}, \frac{\pi}{6}\pi, \frac{5}{6}\pi, \frac{7}{6}\pi; \frac{\pi}{3}, \frac{5}{3}\pi, \frac{2}{3}\pi, \frac{4}{3}\pi; \frac{1}{2}\pi, \frac{3}{2}\pi$.

[*E. L. Sherwood.*]

III. From equation (2), we find $\cos\theta=0$; $\pm\frac{1}{2}\sqrt{3}$, or $\pm\frac{1}{2}$. $\therefore\ \theta=\frac{\pi}{2}$,

$\frac{\pi}{6}$, $\pi\pm\frac{\pi}{6}$, $\pm\frac{\pi}{3}$, $\pi\pm\frac{\pi}{3}$; or in *general*, $n\pi\pm\frac{\pi}{2}$, $2n\pi\pm\frac{\pi}{6}$, $(2n+1)\pi\pm\frac{\pi}{6}$, $2n\pi\pm\frac{\pi}{3}$, $(2n+1)\pi\pm\frac{\pi}{3}$. [*J. B. Faught.*]

IV. From equation (2), $\cos\theta=0$. $\therefore$ $\theta=90°$, $270°$,.... If $x=\cos^2\theta$, the second factor becomes $x^2-x=-\frac{3}{16}$, and $x_1=\frac{3}{4}$, $x_2=\frac{1}{4}$. $\therefore$ $\cos\theta=\frac{1}{2}\sqrt{3}$ and $\cos\theta=\frac{1}{2}$. $\therefore$ $\theta=30°$, $330°$...., and $\theta=60°$, $300°$...., for *particular* values. [*J. C. Corbin.*]

V. Contracting the last two terms of (1) into a *product of cosines*, by the formula $\cos x+\cos y=2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$, we have $\cos\theta+2\cos4\theta\cos\theta=0$, or $\cos\theta(1+2\cos4\theta)=0$, $\therefore$ $\cos\theta=0$, and $\theta=\frac{\pi}{2}$ or $90°$, or in general $\frac{1}{2}(2n\pm1)$; also, $1+2\cos4\theta=0$, $\therefore$ $\cos4\theta=-\frac{1}{2}$, and $\theta=30°$, $60°$, or in general $\frac{1}{6}(3n\pm1)$. [*G. I. Hopkins, Otto Clayton.*]

VI. Equation $(1)=\dfrac{\cos3\theta\sin3\theta}{\sin\theta}=0$. $\therefore$ $\cos3\theta=0$....(a), or, $\sin3\theta=0$....(b). From (a), $\theta=\frac{1}{6}(2n+1)\pi$; from (b), $\theta=\frac{1}{3}n\pi$.
[*O. W. Anthony.*]

VII. $\cos3\theta=\cos\theta\cos2\theta-\sin\theta\sin2\theta$; $\cos5=\cos\theta\cos4\theta-\sin\theta\sin4\theta$. $\sin\theta=\sqrt{1-\cos^2\theta}$, $\sin2\theta=2\cos\theta\sqrt{1-\cos^2\theta}$; $\sin4\theta=4\cos\theta(2\cos^2\theta-1)\sqrt{1-\cos^2\theta}$, $\cos2\theta=2\theta=2\cos^2\theta-1$. Substituting, etc., the given equation becomes $\cos4\theta-\cos^2\theta=-\frac{3}{16}$; whence $\cos\theta=\frac{1}{2}$ or $\frac{1}{2}\sqrt{3}$, giving $\theta=60°$ or $30°$ for *particular* values. [*A. H. Bell.*]

NOTE.—The particular values given for this problem in Bowser's *Treatise on Trigonometry*, page 128, are $\frac{\pi}{2}$, $\frac{2}{3}\pi$.—*Editor.*

Also solved by *P. S. BERG, A. L. FOOTE, F. P. MATZ, P. H. PHILBRICK, J. F. W. SCHEFFER C. D. SCHMITT, W. I. TAYLOR*, and *G. B. M. ZEER.*

47. Proposed by LEONARD E. DICKSON, A. M., Fellow in Mathematics, University of Chicago, Chicago, Illinois.

Prove that $(-1)(-1)=+1$.

Solutions by the PROPOSER; G. B. M. ZEER, A. M., Ph. D., Professor of Mathematics in Inter State College, Texarkana, Texas; H. W. DRAUGHON, Olio, Mississippi; P. H. PHILBRICK, M. S., Chief Engineer for Kansas City, Watkins & Gulf Railway Co., Pineville, Louisiana; J. H. GROVE, Professor of Mathematics in Howard Payne College, Brownwood, Texas; P. S. BERG, Apple Creek, Ohio; W. I. TAYLOR, Baldwin University, Berea, Ohio; Professor J. F. W. SCHEFFER, A. M., Hagerstown, Maryland; A. L. FOOTE, C. E., Middlebury, Connecticut; and F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

I. *Assuming* the distributive law to hold, $(-1)\{(+1)+(-1)\}$, or 0,

$=(-1)(+1)+(-1)(-1)$. Assuming the commutative law, $(-1)(+1)=(+1)(-1)=-1$. $\therefore -1+(-1)(-1)=0$, or $(-1)(-1)=+1$.

This proof was suggested by a longer one due to Professor D. A. Hull of Upper Canada College [*L. E. Dickson.*]

II. $(-1)(-1)$ means that -1 is to be taken subtractively one time. $\therefore 0-(-1)=+1$. $\therefore (-1)(-1)=+1$. [*G. B. M. Zerr.*]

III. $-1\times a=-a$. $-1\times(a-1)=-(a-1)=-a+1$. $\therefore -1\times[(a-1)-a]=-a+1-(-a)=-a+1+a=1$ [*P. H. Philbrick.*]

IV. $(-1)(-1)=(-1)(+1)-(-1)(+2)=-1-(-2)=-1+2=+1$. [*H. W. Draughon.*]

V. Definition: $-n$ is the number which added to $n=0$. We know that $(-1)\times 1=-1$; suppose $(-1)\times(-1)=x$. Adding we get, $(-1)(1-1)=x-1$. But $(-1)(1-1)=0$. $\therefore x-1=0$. $\therefore x=+1$. $\therefore (-1)(-1)=+1$. [*J. H. Grove.*]

VI. To multiply one number by another we do to the first what is done to unity to produce the second. [See Smith's *Algebras*, Van Velzer and Slichter's *Univ. Alg.*] $\therefore (-5)(-3)=(-5)(-1-1-1)=-(-5)-(-5)-(-5)=+5+5+5=15$. Similarly, $(-1)(-1)=(-1)(-1)=-(-1)=+1$. [*P. S. Berg, F. P. Matz.*]

VII. According to Wood's *Elements of Algebra, 17th edition*, we have $(-5)(-3)=+15$. Here -3 is to be subtracted 5 times; that is, -15 is to be *subtracted*. Now, subtracting -15 is the same as adding $+15$. Therefore, we have to *add* $+15$. Similarly, $(-1)(-1)=+1$. [*W. I. Taylor, F. P. Matz.*]

VIII. The case $(-a)(-b)=+ab$ is purely conventional and consequently an assumption, which, however, does not deprive the result of its great importance to algebraic operations. [*J. F. W. Scheffer.*]

IX. For illustration, $(a-b)\times(c-b)=(c-b)a-(c-b)b$, but $(c-b)a=ac-ab$ and $(c-b)b=cb-b.b$, and we have $(a-b)(c-b)=ac-ab-(cb-bb)$. Now as we are to take cb less $b.b$ from $ac-ab$, we first take cb and we have $ac-ab-cb$; which is too much by $b.b$; we therefore add $b.b$ and get $ac-ab-cb+b.b$, but $b.b$ is found from $(-b)(-b)=+b.b$. Take $b=1$, then $(-1)(-1)=+1$. [*A. L. Foote.*]

X. Revolve the vector $(+a)$ about its origin A, through an angle of 180°, and it will become the vector $(-a)$, or will be *multiplied* by (-1). Making a equal to unity, then revolving the vector (-1) about its origin A, through an angle of 180°, and it will become the vector (+1), or will be multiplied by (−1); that is, (−1)(−1)=+1. [*F. P. Matz.*]

PROBLEMS.

56. Proposed by CHAS. E. MYERS, Canton, Ohio, and Hon. JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

(a). How much can be paid for a bond, bearing 5% interest and having ten years to run, so as to realize 3% on the investment? [*C. E. Myers*]; (b), At what price must the government sell 5% $100 bonds to run ten years, interest payable annually, to make them the same to the buyer as 3% bonds at par, to run ten years, interest payable anually, provided the buyer can invest all interest received at 4% interest payable anually? [*J. H. D.*]

57. Proposed by J. C. CORBIN, Pine Bluff, Arkansas.

Find the quotient of

$$\begin{vmatrix} (s-a_1)^2 & a_1^2 & a_1^2 \ldots\ldots & a_1^2 \\ a_2^2 & (s-a_2)^2 & a_2^2 \ldots\ldots & a_2^2 \\ a_3^2 & a_3^2 & (s-a_3)^2 \ldots\ldots & a_3^2 \\ \ldots & \ldots & \ldots & \ldots \\ a_n^2 & a_n^2 & a_n^2 \ldots\ldots & s-a_n^2 \end{vmatrix} \div \begin{vmatrix} s-a_1 & a_1 & a_1 \ldots\ldots & a_1 \\ a_2 & s-a_2 & a_2 \ldots\ldots & a_2 \\ a_3 & a_3 & s-a_3 \ldots\ldots & a_3 \\ \ldots & \ldots & \ldots & \ldots \\ a_n & a_n & a_n \ldots\ldots & s-a_n \end{vmatrix}$$

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

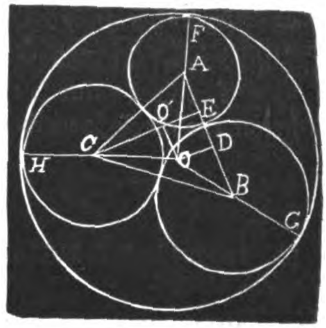
45. Proposed by B. F. BURLESON, Oneida Castle, New York.

Determine the radius of a circle circumscribing three tangent circles of a radii $a=15$, $b=17$, and $c=19$.

I. Solution by the PROPOSER; J. F. W. SCHEFFER, A. M., Hagerstown, Maryland; A. H. BELL, Hillsboro, Illinois; and F. P. MATZ, M. Sc., Ph. D., Mechanicsburg, Pennsylvania.

The problem has two cases: first, when the three given circles are tangent internally to the required circle, as in the problem; and, second, when the required circle is tangent to them externally. But *one solution* involving the resolution of a quadratic equation, will give the answers to both cases. We give the figure for the *first* case only.

Join the centers of the three given circles forming the triangle ABC. Put $AF=a=15$ ft., $CH=b=17$ ft., and $BG=c=19$ ft. Draw CE perpendicular to AB. Let O be the center of the required fourth tangent circle. Draw the radii $R=OCH=OAF=OBG$. Drop on CE the perpendicular OO', and on AB the perpendicular OD. We have $AC=a+b$, $AB=a+c$ and $BC=b+c$. It is evident that $AO=R-a$, $BO=R-c$, and $CO=R-b$. We have in the triangle ABC, by geometry: $AB:BC+AC::BC-AC:BE-AE$; that is, $c+a:a+2b+c::c-a:(a+2b+c)(c-a)=BE-AE$.



∴ $AE=(a^2+ac+ab-bc)\div(c+a)$, and $BE=(c^2+ac+bc-ab)\div(c+a)$. Again in the triangle AOB, we have $AB:AO+BO::AO-BO:AD-BD$; that is, $c+a:2R-(c+a)::c-a:(2R-a-c)(c-a)\div(c+a)=AD-BD$. ∴ $AD=(a^2+ac+Rc-Ra)\div(c+a)$, and $BD=(c^2+ac-Rc+Ra)\div(c+a)$. Now $OO'=DE=BE-BD=(bc-ab+Rc-Ra)\div(c+a)$; also $EO'=DO=\sqrt{(BO^2-BD^2)}=2\sqrt{(R^2ac-Rac^2-Ra^2c)}\div(c+a)$. The perpendicular $CE=2\sqrt{[abc(a+b+c)]}\div(c+a)$. Now $CO'=CE-EO'=\{2\sqrt{[abc(a+b+c)]}-2\sqrt{(R^2ac-Rac^2-Ra^2c)}\}\div(c+a)$. Again $CO=\sqrt{(CO'^2+OO'^2)}=\sqrt{\{4a^2bc+4ab^2c+4abc^2+4R^2ac-4Rac^2-4Ra^2c+(bc-ab+Rc-Ra)^2-8\sqrt{[abc(a+b+c)(R^2ac-Rac^2-Ra^2c)]}\}}\div(c+a)$. Putting this value of $CO=R-b$ and clearing the resulting equation from radicals by two successive involutions, we obtain the quadratic equation, after dividing by $(c+a)^2$: $[2abc(a+b+c)-(a^2b^2+a^2c^2+b^2c^2)]R^2-2abc(ab+ac+bc)R=a^2b^2c^2$.

Whence by resolution,

$$R=\frac{abc(ab+ac+bc)\pm2abc\sqrt{abc(a+b+c)}}{2abc(a+b+c)-(a^2b^2+a^2c^2+b^2c^2)}=\frac{4181235\pm4941901\sqrt{95}}{243611}$$

$=36.93594828+$ feet, or $-2.60880378+$ ft. The negative value of R, regarded as a positive quantity, is the radius of the circle that is tangent to the three given circles *externally*.

II. Solution by G. B. M. ZERR, A. M., Ph D., Vice President and Professor of Mathematics in the Texarkana College, Texarkana, Arkansas.

Let $BG=a,=19$, $CH=b=17$, $AF=c=15$, $OG=OH=OF=r$.

$$\therefore \cos BCA=\frac{b^2+ab+bc-ac}{b^2+ab+bc+ac}=\frac{97}{192},$$

$$\cos BCO=\frac{b^2+ab+ar-br}{ar+br-b^2-ab}=\frac{306+r}{18r-306},$$

$$\cos ACO=\frac{b^2+bc+cr-br}{cr+br-b^2-bc}=\frac{272-r}{16r-272}.$$

But $\cos BCA=\cos(BCA+BCO)$.

$$\therefore r=\frac{abc\{ab+ac+bc\pm2\sqrt{abc(a+b+c)}\}}{4abc(a+b+c)-(ab+ac+bc)^2}=\frac{\{863\pm102\sqrt{95}\}4845}{243611};$$

$\therefore r=36.93595$ or -2.608803.

The first value is the radius of circumscribing circle, the second is the radius of the circle inscribed in the space enclosed by the three circles.

PROBLEMS.

50. **Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.**

Draw a line perpendicular to the base of a triangle dividing the triangle in the ratio of m to n.

51. **Proposed by G. B. M. ZERR, A. M., Ph. D., Vice President and Professor of Mathematics in the Texarkana College, Texarkana, Arkansas.**

To construct a trapezoid; given the bases, the perpendicular distance between the bases and the angle formed by the diagonals.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him

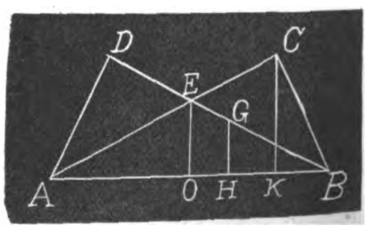
SOLUTIONS OF PROBLEMS.

36. **Proposed by H. C. WHITAKER, B. Sc., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.**

A cube is revolved on its diagonal as an axis. Define the figure described and calculate its volume.

Solution by G. B. M. ZERR, A. M., Ph. D., Vice President and Professor of Mathematics in the Texarkana College, Texarkana, Arkansas.

The surface of the solid is that generated by revolving $ADECB$ about AB as an axis.



Let $AD=BC=a$ be the side of the cube. Then $AC=BD=a\sqrt{2}$, $AB=a\sqrt{3}$, $OC=\frac{a\sqrt{3}}{2}$. $AC^2=AB.AK$. $\therefore AK=\frac{2a}{\sqrt{3}}$.

$CK^2=AK.BK=AK(AB-AK)$.

$$\therefore CK=\frac{a\sqrt{2}}{\sqrt{3}}.$$

$$GH=\tfrac{1}{3}CK=\frac{a\sqrt{2}}{3\sqrt{3}}.\quad EO:AO=CK:AK.\quad \therefore EO=\frac{a\sqrt{6}}{4}.$$

Area ACB=area $ADB=\frac{1}{2}AB.CK=\frac{a^2\sqrt{2}}{2}=A$.

Area $AEB=\frac{1}{2}AB.EO=\frac{3a^2\sqrt{2}}{8}=A'$.

V=required volume$=2(2\pi GH.A)-2\pi EO.\frac{A'}{3}$.

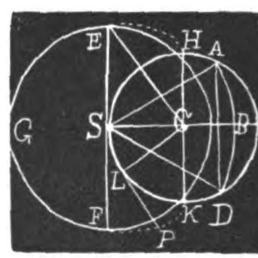
$\therefore\ V=\frac{4a^3\pi}{3\sqrt{3}}-\frac{a^3\pi\sqrt{3}}{8}=\frac{23a^3\pi}{24\sqrt{3}}=\frac{23a^3\pi\sqrt{3}}{72}$.

37. Proposed by J. A. CALDERHEAD, Superintendent of Schools, Limaville, Ohio.

A man ties two mules—one to the outside of a circular wall, the other to the inside. If the lengths of the ropes of each is one-fourth the circumference of the wall, and both together can graze over one acre of ground: find the circumference of the wall.

I. Solution by COOPER D. SCHMITT, Professor of Mathematics in the University of Tennessee, Knoxville, Tennessee.

Let S be the point where the mules are fastened. The mule grazing on the outside, grazes over the semi-circle $EGF+HSK$ and FSP. The mule on the inside grazes over the segment SHA+ segment SKD+ sector ASD.



We must find the area of these different portions and their sum equals one acre or 160 sq. rods.

1. To find the area of sector $ABDSA$. Since $AS=\frac{\pi r}{2}$, we have $\cos ASB=\frac{1}{4}\pi$. $\therefore$ $ASB=38°\ 15'$, and area of sector $=\frac{76\frac{1}{2}}{360}.\pi.\frac{\pi^2r^2}{4}$.

2. To find area of segment SHA. $\angle SCA=103°\ 30'$. Segment SHA=sector $SHAC-\triangle ASC$ $=\frac{103\frac{1}{2}}{360}\pi r^2-\frac{1}{2}r^2\sin 103\frac{1}{2}°$. Segment DSK=segment SAH.

3. Area of $EGF=\frac{1}{2}\pi\left(\frac{\pi^2r^2}{4}\right)$.

4. Area of involute $ESHA$=area CHE+area ECS−area SCH =(by the Calculus)$\frac{SC^3}{6r}+\frac{1}{2}SC.r-\frac{1}{4}\pi r^2$. But $SC=\frac{1}{2}\pi r$. $\therefore$ involute $ESHA=(\frac{1}{2}\pi r)^3\div 6r+\frac{1}{2}.\frac{1}{2}\pi r.r-\frac{1}{4}\pi r^2=\frac{1}{48}\pi^3r^2$.

The area of involute $FSLK$=area of involute $ESHA$.

Combining these different areas, we have

$$r^2\left[\frac{153}{720}\frac{\pi^3}{4}+\frac{207}{360}\pi-\sin 103\tfrac{1}{2}°+\frac{\pi^3}{8}+\frac{n^3}{24}\right]=160.$$

$$r^2\left[\frac{51\pi^3}{960}+\frac{207}{360}\pi-.9724+\frac{\pi^3}{6}\right]=160,$$

$$r^2\left[\frac{211}{960}\pi^2+\frac{207}{360}\pi-.9724\right]=160,$$

$r^2[6.8148+1.8064-.9726]=160$, $r^2(7.6486)=160$, $r^2=20.91$, $r=4.57$. $2\pi r=28.714+$ the circumference of the wall.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Vice President and Professor of Mathematics in the Texarkana College, Texarkana, Arkansas.

Let ASD, in Fig. above, be the given wall. S the point where the mules are fastened, a=radius of wall, $\varphi=\angle KCL$, $PL=\rho$, =radius of curvature of involute KPF, $\theta=\angle ASC$.

We now have the three areas to find:— (1) area $SHABDKS$, (2) the the two equal involute areas SHE and SKF, (3) area semi-circle EGF. $SA=SE=\frac{1}{2}\pi a$.

$\therefore$ Area semi-circle $EGE=\frac{1}{8}\pi^3a^2\ldots.(1)$.

Area of an element between two consecutive radii of curvature is $dA=\frac{1}{2}\rho^2d\varphi=\frac{1}{2}a^2\varphi^2d\varphi$, since $\rho=a\varphi$.

$$\therefore \text{ Area } (SHE+SKF)=a^2\int_0^{\frac{1}{2}\pi}\varphi^2d\varphi=\tfrac{1}{24}\pi^3a^2\ldots.(2).$$

Area common to both circles$=a^2(\pi+2\theta\cos2\theta-\sin2\theta)$, but $2a\cos\theta=\frac{1}{2}a\pi$, $\therefore$ $\cos\theta=\frac{1}{4}\pi$.

$\therefore$ Area common to both circles

$$=a^2\left\{\pi+\tfrac{1}{4}(\pi^2-8)\cos^{-1}\frac{\pi}{4}-\frac{\pi}{8}\sqrt{16-\pi^2}\right\}\ldots.(3).$$

$$\therefore a^2\left\{\tfrac{1}{6}\pi^3+\pi+\tfrac{1}{4}(\pi^2-8)\cos^{-1}\frac{\pi}{4}-\frac{\pi}{8}\sqrt{15-\pi^2}\right\}=160 \text{ sq. rods.}$$

$$\therefore a^2\left(7.337+.4674\cos^{-1}\frac{\pi}{4}\right)=160 \text{ sq. rods.}$$

$\therefore$ $7.64896a^2=160$, $a=4.5736$ rds.

$2\pi a=28.7368$ rods, =circumference required.

Also solved by *A. H. BELL* and *F. P. MATZ.*

PROBLEMS.

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45. Proposed by Dr. GEORGE LILLEY, Portland, Oregon.

A fly starts from a point in the circumference of a table, 3 feet in diameter. and travels uniformly along the diameter to a point in the circumference of the table directly opposite the starting point. The table moves uniformily to the right about a center axis in such manner that it makes one complete revolution while the fly passes over its diameter. Find the absolute path described by the fly and the ratio of rates of movement of the table and the fly.

46. Proposed by H. C. WHITAKER, M. E., Sc. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

There are four points A, B, C, and D in space. Point D remains fixed with its co-ordinates (1, 2, 2) feet. At a given time A is at (2, 3, 4) feet, is moving in a straight line at the rate of 3 feet per minute, and has passed through (5, 9, 10) feet: B is at (1, 4, 2) feet, moves in a straight line at the rate of 7 feet per minute, and will pass through (−2, 2, 8) feet; C is at the origin and moves along the axis of X in the direction of x positive at the rate of 6 feet per minute.

The motion of the points being continuous before and after the given time, required the times when the volume of the tetrahedron whose edges are the lines joining these points will be 108 cubic inches.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

26. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

If an elastic sphere be electrified in such a manner that the initial internal pressure remains constant, determine an expression for the *ratio of the electrical densities* when the volume of the sphere has been increased to $(m+1)$ times its initial volume.

I. Solution by the PROPOSER.

Assuming r as the *initial* radius, and R as any *greater* radius, we have $x^2+y^2+z^2=r^2$ ....(1) and $x^2+y^2+z^2=R^2$ ....(2), as the result of the *volume-increase*. These Cartesian equations are transformable by means of the well-known equations, $x=\rho\cos\theta$ and $y=\rho\sin\theta$. Choosing the integral limits so as to give the total internal pressure, we have for the work done in increasing the initial sphere, radius r, to a sphere of radius R, when $z_1=\sqrt{(r^2-\rho^2)}$ and $z_2=\sqrt{(R^2-\rho^2)}$,

$$W=8\left(\frac{2T}{r}\right)\int_0^{\frac{1}{2}\pi}\int_0^r 2\rho d\theta d\rho\times 8\int_0^{\frac{1}{2}\pi}\left[\int_0^R\int_0^{z_2}\rho d\rho dz-\int_0^r\int_0^{z_1}\rho d\rho dz\right]d\phi$$

$$=\tfrac{4}{3}\pi(8\pi Tr)[R^3-r^3]\ldots.(3).$$

Specializing in (3), for $R^3=mr^3$ and for $R^3=(m+1)r^3$; representing the electrical densities, the ratio of which is to be determined, by Δ_m and Δ_{m+1}; then equating the specialized results obtained from (3), to the potential energies of the electrifications similarly specialized,—that is, according to Helmholtz's formula, to

$$\frac{Q}{2r}=\left(\frac{\Delta h}{2r}\right)\left[8\int_0^{\frac{1}{2}\pi}\int_0^r 2\rho d\theta d\rho\right]^2,$$

we have the required ratio,

$$\frac{\Delta_m}{\Delta_{m+1}}=\sqrt{\left(1-\frac{1}{m^2}\right)}\ldots.(4).$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Vice President and Professor of Mathematics in the Texarkana College, Texarkana, Arkansas.

Let p be the excess of pressure, on the assumption that the external pressure is costant.

Let u=original volume, then when the sphere is n times its original volume, the work done in this electrification must be

$p(n-1)u$, (see *Minchin's Statics*).

Let v be the potential, Q the charge of this electrification, r=radius of sphere, σ=electrical density.

$\therefore$ energy of electrification$=\frac{1}{2}vQ$. But $v=\dfrac{Q}{r}$ and $Q=4\pi r^2\sigma$.

$\therefore p(n-1)u=\frac{1}{2}\dfrac{Q^2}{r}=8\pi^2r^3\sigma^2$. But $\frac{4}{3}\pi r^3=nu$.

$\therefore p(n-1)u=6n\pi\sigma^2u$.

$\therefore p(n-1)=6n\pi\sigma^2$.

Similarly, when the sphere becomes $(m+1)$ times its original size we get, if σ_1 is the density, $pm=6(m+1)\pi\sigma_1^2$.

$$\therefore \frac{\sigma}{\sigma_1}=\sqrt{\frac{(m+1)(n-1)}{mn}}=\sqrt{\frac{m^2-1}{m^2}} \text{ if } n=m.$$

27. Proposed by G. B. M. ZERR, A. M., Ph. D., Vice President and Professor of Mathematics in the Texarkana College, Texarkana, Arkansas.

One thousand balls, each having a mass of 10 grams, and each moving with a velocity of 10 kilometers per second, are confined in a certain space with elastic walls. Into the same space are now introduced one thousand balls each of 1000 grams mass, and moving with a velocity each of 10 kilometers per second; collisions take place, and finally, after a number of encounters, the average kinetic energy of each of the two thousand balls is the same. Show that this is 5.75(10)11 in the centimeter-gram-system.

Solution by O. W. ANTHONY, Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland and the PROPOSER.

By Avogadro's hypothesis and kinetic theory, if M, M_1 are the masses and V, V_1 the velocities of two balls, then $\frac{1}{2}MV^2$ and $\frac{1}{2}M_1V_1^2$ are their respective kinetic energies. As the average kinetic energy of each ball is equal, we get $\frac{1}{2}MV^2=\frac{1}{2}M_1V_1^2$.

$\therefore$ The average kinetic energy$=\frac{1}{2}(\frac{1}{2}MV^2+\frac{1}{2}M_1V_1^2)=E$.

$\therefore E=\frac{1}{4}(MV^2+M_1V_1^2)$.

Let $V=V_1$, then $E=\frac{1}{4}(M+M_1)V^2$.

V=10 kilometers$=10^6$ centimeters.

M=10 grams, $M_1=100$ grams.

$\therefore E=\frac{1}{4}(10+100)10^{12}=\frac{11}{4}\times10^{13}=2.75\times10^{13}=2.75(10)^{13}$.

Also solved by *F. P. MATZ.*

PROBLEMS.

33. **Proposed by F P. MATZ, M. Sc , Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania**

At what angle with the axis of a stalk must a *sharp* wedge-shaped blade be struck, in order to sever the stalk with the *least* force?

34. **Proposed by O. W. ANTHONY, Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland**

A particle is placed within a thin cylindrical shell without ends. Find the resultant attraction, the cylinder being composed of matter attracting according to the laws of nature.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

25. **Proposed by G. B. M. ZERR, A. M., Ph. D., Vice President and Professor of Mathematics in Texarkana College, Texarkana, Arkansas.**

The probability that the distance of two points taken at random in a given convex area, A, shall excel a given limit (a) is

$$\Delta=\frac{1}{3A^2}\int\int(C^3-3aC+2a^3)dpd\theta,$$

where C is a chord of the area, whose co-ordinates are $p\ \theta$; the integration extending to all values of p, θ, which give a chord $C>a$. What is Δ when the area is a circle? If in the circle $a=r=$radius, $\Delta=\frac{3\sqrt{3}}{4\pi}$.

Solution by the PROPOSER.

Let ST be the chord; P, Q the random points, $OR=p$, the perpendicular, $\angle ROA=\theta$, $SQ=x$, $PQ=y$, $ST=C$.

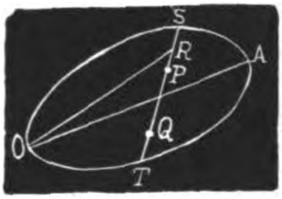
An element of area at Q is $dpdx$; at P, $yd\theta dy$. The limits of x are o and $C-a$; of y, $C-x$ and a, and doubled.

Since two points can be taken in A^2 ways in the area A, we get for the required chance

$$\Delta = \frac{2}{A^2}\int\int d\theta dp \int_0^{C-a}\int_a^{C-x} y dx dy,$$

$$= \frac{1}{A^2}\int\int d\theta dp \int_0^{C-a}(C^2-2Cx+x^2-a^2)dx,$$

$$= \frac{1}{3A^2}\int\int (C^3-3a^2C+2a^3)d\theta dp.$$



Now let the area be a circle with the origin at centre. Then $C=2\sqrt{r^2-p^2}$, when r=radius. The limits of θ are 0 and $\frac{1}{2}\pi$, doubled, of p, 0 and $\frac{1}{2}\sqrt{4r^2-a^2}$, and doubled.

$$\therefore\ \Delta = \frac{4}{3\pi^2 r^4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\sqrt{4r^2-a^2}}\left\{8(r^2-p^2)^{\frac{3}{2}}-6a^2(r^2-p^2)^{\frac{1}{2}}+2a^3\right\}d\theta dp.$$

$$= \frac{1}{2\pi^2 r^4}\left\{(a^3+2ar^2)(4r^2-a^2)^{\frac{1}{2}}+8r^2(r^2-a^2)\sin^{-1}\left(1-\frac{a^2}{4r^2}\right)^{\frac{1}{2}}\right\}\int_0^{\frac{1}{2}\pi}d\theta$$

$$= \frac{1}{4\pi}\cdot\frac{a}{r}\left(2+\frac{a^2}{r^2}\right)\left(4-\frac{a^2}{r^2}\right)^{\frac{1}{2}}+\frac{2}{\pi}\left(1-\frac{a^2}{r^2}\right)\sin^{-1}\left(1-\frac{a^2}{4r^2}\right)^{\frac{1}{2}}.$$

If $a=r$, $\Delta=\frac{3\sqrt{3}}{4\pi}$.

26. Proposed by J. WATSON, Middlecreek, Ohio.

Find the average area of all right-angled triangles having a given hypotenuse.

I. Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Let h=the given hypotenuse, and x=the base; then will $\sqrt{(h^2-x^2)}$= the perpendicular, and the area of the triangle is $A=\frac{1}{2}x\sqrt{(h^2-x^2)}$. Hence the required average area becomes, if $\frac{1}{2}h\sqrt{2}=a$,

$$A=\int_0^a A dx \div \int_0^a dx, = \tfrac{1}{12}h^2(2\sqrt{2}-1).$$

Second Solution.

Represent the base by $h\cos\theta$, and the perpendicular by $h\sin\theta$; then

we have $A=\frac{1}{2}h^2\int_0^{\frac{1}{4}\pi}\sin\theta\cos^2\theta d\theta+h\int_0^{\frac{1}{4}\pi}\cos\theta d\theta, = \frac{1}{12}h^2(2\sqrt{2}-1).$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Vice President and Professor of Mathematics in Texarkana College, Texarkana, Arkansas; O. W. ANTHONY, Professor of Mathematics, New Windsor College, New Windsor, Maryland; J. F. W. SCHEFFER, A. M., Hagerstown, Maryland; and H. W. DRAUGHON, Olio, Mississippi.

Let $AC=2a$=hypotenuse of triangle, $AD=DC=DB=a$, and $\angle CDB=\theta$. $\therefore BE=a\sin\theta$.

$\therefore$ Area $=a^2 \sin\theta$. Perimeter$=2a(\cos\frac{1}{2}\theta+\sin\frac{1}{2}\theta+1$.

Let A=average area, P=average perimeter.

$$\therefore A=\frac{a^2\int_0^\pi \sin\theta d\theta}{\int_0^\pi d\theta}=\frac{2a^2}{\pi}.$$

$$P=\frac{2a\int_0^\pi(\cos\frac{1}{2}\theta+\sin\frac{1}{2}\theta+1)d\theta}{\int_0^\pi d\theta}=\frac{2a(4+\pi)}{\pi}.$$

III. Solution by O. W. ANTHONY, Professor of Mathematics, New Windsor College, New Windsor, Maryland; P. S. BERG, Larimore, North Dakota; and H. W. DRAUGHON, Olio, Mississippi.

Let a=the hypotenuse, and x one of the sides.

Then the area of the triangle $=\frac{1}{2}\times\sqrt{(a^2-x^2)}$ and the required average area

$$=\frac{\int_0^a \frac{1}{2}x\sqrt{(a^2-x^2)}}{\int_0^a dx}=\frac{a^2}{6}.$$

NOTE.—We have published these various solutions in order that the authors may compare their results and decide upon some definite method of solving this problem. It is our opinion that the result, $\frac{a^2}{2\pi}$, is correct; for the number of triangles is equal to the semi-circumference whose diameter is the given hypotenuse a, that is to say, the number of triangles is proportional to the locus of the vertex of the right angle and not proportional to the variable sides. But if this method of solution is adopted for this problem, it will vitiate the solutions of a great many problems in *Average* and *Probability*,—solutions that have gone in print in numerous Journals and text books.

Dr. Artemas Martin proposed this problem in the *Educational Times*, London, England, for October, 1869. The published solutions both in the *Times* and the *Reprint* give the answer $\frac{a^2}{2\pi}$. Dr. Martin says, *Mathematical Magazine*, Vol I., p. 216, "I do not regard that method [the method assuming that the vertices of the right angle are uniformly distributed on the semi-circumference of a circle whose diameter is a] as correct. The vertices of the right angle will all be situated on a semi-circumference whose diameter is a, but they will *not* be uniformly distributed on it. In order to obtain *all* the triangles, one of the legs should be made to vary uniformly from 0 to a."

He then produces a very beautiful solution without the aid of the calculus and gets as a result, $\frac{1}{6}a^2$. Then he gives another solution which is the same as III. above.

Now it seems to us that whether the triangles are uniformly distributed on the semi-circumference or not is of no concern in the solution of the problem. The question is (1), how many right triangles are there whose hypotenuses are a; and (2), what is the area of each one of these triangles? Having found the numbers answering to these questions, we divide the sum of the areas of the triangles by the number of triangles, according to the principle of *Mean Value*, and get the required result. The *sum* of the areas of the triangles is easily found by the aid of the Calculus and the number of triangles is equal to the semi-circumference of a circle whose diameter is a. This is, in our opinion, the correct solution and agrees with II. above. All of the above solutions are, doubtless, correct from the stand-points of the authors, but the stand-points of some must be wrong. As it is the object of the MONTHLY to aid in the establishment of sound principles in all departments of Mathematics, we shall be pleased to publish, in the next issue, brief notes on these solutions from various contributors. [EDITOR.]

PROBLEMS.

33. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the average area of all regular polygons having a *constant* apothem.

34. Proposed by B. F. FINKEL, A. M., Professor of Mathematics, Drury College, Springfield, Missouri.

Two points are taken at random on the circumference of a semi-circle. Find the chance that their ordinates fall on either side of a point taken at random on the diameter.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

DIOPHANTUS' EPITAPH.

Hic Diophantus habet tumulum, qui tempora vitae
Illius mira denotat arte tibi,
Egit sextantem juvenis; languine malas
Vestire hinc coepit parte duodecima.

Septante uxori post haec sociatur, et anno
Formosus quinto nascitur inde puer.

Semissem aetatis postquam attigit ille paternae
Infelix subita morte peremptus obit.
Quatuor aestates genitor lugere superstes
Cogitur: hinc annos illius assequere.

An Equation for the "Sum of Squares equal a Square" by R. J. ADCOCK, Larchland, Illinois.

The following identical equation for the sum of squares=a square, I have not seen published. If $u=x+y+z+v+w$, $u^2=x^2+y^2+z^2+v^2+w^2+2xy+2xz+2xv+2xw+2yz+2yv+2yw+2zv+2zw+2vw$; and if the sum of products two in a set=0, $u^2=x^2+y^2+z^2+v^2+w^2$, $w=-\frac{xy+xz+xv+yz+yv+zv}{x+y+z+v}$,

$$u^2=x^2+y^2+z^2+v^2+\left(\frac{xy+xz+xv+yz+yv+zv}{x+y+z+v}\right)^2=$$

$$\left[x+y+z+v-\left(\frac{xy+xz+xv+yz+yv+zv}{x+y+z+v}\right)\right]^2.$$

Clearing of fractions and reducing, $[x(x+y+z+v)]^2+y^2(x+y+z+v)^2+z^2(x+y+z+v)^2+v^2(x+y+z+v)^2+(xy+xz+xv+yz+yv+zv)^2=(x^2+y^2+z^2+v^2+xy+xz+xv+yz+yv+zv)^2$. True for three or any greater number of letters.

COMMENT.—In the solution of problem 21, page 163, Vol. II, May No., Dr. Martin uses an ingenuous method for finding a general formula "to find nine integral square numbers whose sum is a square number."

The same formula, expressed for finding n integral square numbers whose sum is a square number, may be produced, more directly, from $(2pq)^2+(p^2-q^2)^2=(p^2+q^2)^2$. Put $p^2=m_1^2+m_2^2+m_3^2+\ldots\ldots+m^2_{n-1}$ and $q^2=m^2_n$, in which $m_1 m_2, m_3, \ldots\ldots m_n$ represent any n integers.

We readily obtain $(2m_1m_n)^2+(2m_2m_n)^2+(2m_3m_n)^2+\ldots\ldots+(2m_{n-1}m_n)^2+(m_1^2+m_2^2+m_3^2+\ldots\ldots+m^2_{n-1}-m^2_n)^2=(m_1^2+m_2^2+m_3^2+\ldots\ldots+m^2_n)^2$.

Illustration. Let $n=9$, and put $m_1=1$, $m_2=2$, $m_3=3$, $m_4=4$, $m_5=5$, $m_6=6$, $m_7=7$, $m_8=8$, and $m_9=9$. Substituting these values in the formula and dividing by 12, we obtain $1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+14^2=20^2$.

PROBMEMS.

37. Proposed by A. H. BELL, Hillsboro, Illinois.

Find the first four, integral values of n in $\frac{n(5n-3)}{2}=\square$.

This is the general form of septagonal numbers, 1, 7, 18, 34, 55, etc.

38. Proposed by H. C. WILKES, Skull Run West Virginia.

Let n be any number and let $n^3+1=x$. Then $x^3+(2x-3)^3+(nx-3n)^3=n^3x^3$. How can this be demonstrated; it will always be found true on trial.

39. Proposed by F. P. MATZ, M. Sc., Ph. D, Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

The mth root of the nth power of an *integral* number is a perfect pth power. What is the number?

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

16. **Yale Senior Prize Problem.---Contributed by H. A. NEWTON, LL. D., Professor of Mathematics, Yale University, New Haven Connecticut.**

The axes of two right cylinders whose bases are circles of 4 and 6 inches radius respectively, intersect at right angles. Compute to four decimal places the lengths of the *curves* of intersection of the two surfaces.

Solution by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Make $m=6$ inches, and $n=4$ inches; then the Cartesian equations of the cylinders become $z^2+x^2=m^2$ ....(1) and $y^2+x^2=n^2$ ....(2).

$$\therefore \frac{dz}{dx}=-\frac{x}{z}=-\frac{x}{\sqrt{(m^2-x^2)}}\text{....(3)},$$

$$\text{and } \frac{dy}{dx}=-\frac{x}{y}=-\frac{x}{\sqrt{(n^2-x^2)}}\text{....(4)}.$$

Hence the expression for the *lengths* of the *curves* of intersection of the two surfaces becomes, *Todhunter's Integral Calculus*, p. 116,

$$L=8\int_0^n\sqrt{\left(1+\frac{x^2}{m^2-x^2}+\frac{x^2}{n^2-x^2}\right)}dx\text{....(5)}.$$

Make $c^2=n^2/m^2$, and $x=n\sin\phi$; then $dx=n\cos\phi d\phi$. Transforming (5), etc.,

$$L=8\int_0^n\sqrt{\left(\frac{m^2n^2-x^4}{(m^2-x^2)(n^2-x^2)}\right)}dx=8n\int_0^{\frac{1}{2}\pi}\sqrt{\left(\frac{1-c^2\sin^4\phi}{1-c^2\sin^2\phi}\right)}d\phi$$

$$=8n\int_0^{\frac{1}{2}\pi}[1+\tfrac{1}{2}c^2\sin^2\phi+(\tfrac{3}{8}c^4-\tfrac{1}{2}c^2)\sin^4\phi+(\tfrac{5}{16}c^6-\tfrac{1}{4}c^4)\sin^6\phi+\text{etc.}]d\phi$$

$$=4\pi n\left[1+\frac{c^2}{16}+\frac{111c^4}{4016}+\text{etc.}\right]=51.9363+\text{inches}.$$

20. **Proposed by SAMUEL HART WRIGHT, M. D., M. A., Ph. D., Penn Yan, Yates County, New York.**

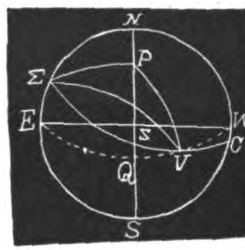
When will the Dog-Star and the Sun *rise together* in lanitude $\lambda=+42°30'$, if the right Ascension of the said star be $\alpha=6h.\ 40m.\ 30s.$ and the Declination $\delta=-16°\ 33'\ 56''$?

Solution by F. P. MATZ, Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

According to *Chauvenet's Spherical and Practical Astronomy*, Vol. I.,

218, Art. 153, we have for the *hour-angle* of the Dog-Star,
$t=\pm\cos^{-1}[-\tan\lambda\tan\delta],=74^\circ\ 10'\ 57.''77,=4h.\ 56m.\ 43.851s.$ The *sidereal* time of the rising of the Dog-star is, therefore, $\Theta=\alpha-t,=1h.\ 43m.\ 46.149s.,$ $=25^\circ\ 56'\ 32''.325.$

The problem now is to find on what day of the year the *Sun* will rise at the sidereal time of the rising of the Dog-Star.



In the accompanying diagram, let Σ represent the Sun just rising; $\angle VPQ=\Theta$; $QZ=\lambda$; $ZP=90-\lambda$; $VP=Z\Sigma=90^\circ$; $\angle PVE=90^\circ$; and $\angle \Sigma VE=23^\circ\,27'\,18''.82$. Now, $\angle PV\Sigma=66^\circ\ 32'\ 41''.18$; and from the spherical triangle PVZ, we deduce $\angle PVZ=25^\circ\,32'\ 24''$, and $VZ=48^\circ\ 30'\ 8''$. Obviously $\angle ZV\Sigma=90^\circ$
$-(\angle PVZ+\angle\Sigma VE)=41^\circ\ 0'\ 17''.18.$
From the spherical triangle $VZ\Sigma$, we deduce $\angle V\Sigma Z=29^\circ\ 26'\ 18''$, and $V\Sigma=131^\circ\ 25'\ 28''.4$. Now, $V\Sigma$ is the *longitude* of the Sun at the time the Dog-Star and the Sun rise together. With a mean daily motion of $59'\ 8''.3302$, the Sun reaches this longitude $133\frac{1}{4}$ days immediately succeeding the *twenty-first* of March, or on the *first* day of August.

This problem can be solved in *four* different ways; and some of the results of these solutions indicate the *second* day of August as the required date.

22. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

From what kind of dry wood must a ship's log be cut, in order that the log may float with its center of gravity at the water's surface?

I. Solution by B. F. BURLESON, Oneida Castle, New York, and the PROPOSER.

The ship's log, a *circular sextant* in form, floats with the *convex* part immersed. If the central semi-angle be α, the portion of the log *not* immersed is αr^2. Consequently, the portion of the log immersed, becomes,

$$A=(\alpha-4\sin^2\alpha\tan\alpha/9\alpha^2)r^2\ \ldots.(1).$$

If $\dfrac{1}{G}$=the specific gravity of the log,

$$\frac{1}{G}=\frac{9\alpha^3-4\sin^2\alpha\tan\alpha}{9\alpha^3}=1-\frac{8\sqrt{3}}{\pi^3}=.5531096,$$

which, according to *Hutton's* Table of Specific Gravity of Bodies, is the specific gravity of Dry *Fir*; and, according to *Scribner's* Table of the Specific Gravity of Bodies, is the specific gravity of Dry *White Pine*.

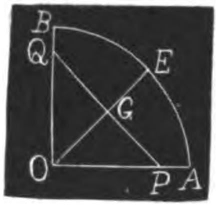
II. Solution by G. B. M. ZERR, A. M., Ph. D., Vice President and Professor of Mathematics and Sciences in the Texarkana College, and Conservatory of Music, Texarkana, Arkansas.

Taking the "log" in the shape of a quadrant of a circle and supposing it to float with the arc down, let G be the center of gravity ρ=density of the wood, 1.026 the density of the sea-water. Then $OG=\frac{4r\sqrt{2}}{3\pi}$, $OP=OQ=\frac{8r}{3\pi}$.

$\therefore$ area $POQ=\frac{32r^2}{9\pi^2}$, area quadrant$=\frac{1}{4}\pi r^2$, area

$QBEAPQ=\frac{1}{4}\pi r^2-\frac{32r^2}{9\pi^2}$, $\therefore 1.026\left\{\frac{1}{4}\pi r^2-\frac{32r^2}{9\pi^2}\right\}$

$=\frac{1}{4}\pi r^2\rho$.



$$\therefore \rho=1.026\left(1-\frac{128}{9\pi^3}\right)=.5554.$$

This is the density of Juniper tree (dry) and very nearly the density of white pine (.554).

23. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics in Texarkana College, Texarkana, Arkansas.

Pliny says, "Thales determined the cosmical setting of the Pleiades to have happened in his time 25 days after the vernal equinox". Determine the time when Thales lived from the following data:—Latitude of Miletus 37° 30′, the precession of the equinox 50″.34 anually, the R. A. of Alcyon (η Tauris) Jan. 1, 1895, 3h. 41m. 15 sec. declination 23° 46′ 49″ N.

Solution by the PROPOSER.

Let λ=latitude of Miletus, a, δ, t, a_1, δ_1, t_1, the R. A. declination and hour-angle of Alcyon and the Sun respectively; ε=the obliquity of the ecliptic, ω=the distance the Sun has traveled on the ecliptic after the vernal equinox.

Then $\cos t=-\tan\lambda\tan\delta$....(1). $\cos t_1=-\tan\lambda\tan\delta_1$....(2). $\sin a_1=\tan\delta_1\cot\varepsilon$....(3). $a_1+t_1=a+t=\theta$, or $a_1=\theta-t_1$....(4). $\sin a_1=\sin(\theta-t_1)$....(5). From (3) and (5), $\sin(\theta-t_1)=\tan\delta\cot\varepsilon$....(6). From (2) and (6), $\tan\delta_1=\frac{\sin(\theta-t_1)}{\cot\varepsilon}=-\frac{\cos t_1}{\tan\lambda}$....(7). From (7)

$$\tan t_1=\frac{\sin\theta\tan\lambda+\cot\varepsilon}{\cos\theta\tan\lambda}\text{....(8).}$$

Also $\cot\omega=\cos\varepsilon\cot a_1$....(9). Now $\lambda=37°\ 30'$, $\delta=23°\ 46'\ 49''$, a=3h. 41m. 15 sec., $\varepsilon=23°\ 27'\ 13''$. From (1), $t=109°\ 45'\ 43''.57$=7h. 19m. 2.9 sec. $a+t=\theta$=11h. 0m. 17.91 sec.$=165°\ 4'\ 28''.57$. From (8) $t_1=106°\ 30'\ 10''.94$ =7h. 6m. 0.73 sec. From (4), a_1=3h. 54m. 17.18 sec.$=58°\ 34'\ 17''.7$. From (9), $\omega=60°\ 43'\ 28''.47$.

In one day the Sun moves 59′ 8″.35. (59′ 8″.35)×25=24°38′ 28″75. 60° 43′ 28″.47−24° 38′ 28″.75=36° 4′ 59″.72=129899″.72. 129899″.72 ÷50″.34=2580.44+years. 2580.44−1894=686.44 B. C., when Thales determined the cosmical setting of the Pleiades. 60° 43′ 28″.47÷59′ 8″.35 =61.6085 days after vernal equinox. 61.6085−25=36.6085. 59′ 8″.35÷50″.34=70.48768 years. 79.48768×36.6085=2580.448+. 2580.438−1894 =686.448 B. C.

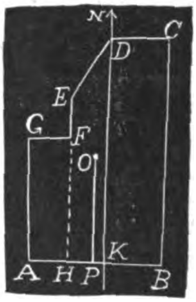
Also solved by *F. P. MATZ.*

24. Proposed by D. H. DAVISON, C. E., Minonk, Illinois.

For the purpose of locating the most eligible point for a county-seat, it is required to determine the centre of a county whose dimensions are as follows: Beginning at the S. W. corner, thence E. 15 miles, thence N. 33¾ miles, thence W. 6 miles to the north end of the meridian running south through the county, thence southwesterly to a point 6 miles W. from the meridian and 9¾ miles S. of the north end of said meridian, thence S 3 miles, thence W. 3 miles, and thence S. 21 miles to the place of beginning.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science in Texarkana College, Texarkana, Arkansas; and F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Let $\bar{x}$, $\bar{y}$, be the co-ordinates of the centroid, and divide the county into three parts as in the figure, then we eaily get with A as origin



$$\bar{x}=\frac{\int_0^3\int_0^{21}xdxdy+\int_3^9\int_0^{19\frac{1}{8}+\frac{13}{8}x}xdxdy+\int_9^{15}\int_0^{33\frac{3}{4}}xdxdy}{\int_0^3\int_0^{21}dxdy+\int_3^9\int_0^{19\frac{1}{8}+\frac{13}{8}x}dxdy+\int_9^{15}\int_0^{33\frac{3}{4}}dxdy}$$

$$=8\tfrac{163}{876}\text{ miles.}$$

$$\bar{y}=\frac{\int_0^3\int_0^{21}ydxdy+\int_3^9\int_0^{19\frac{1}{8}+\frac{13}{8}x}ydxdy+\int_9^{15}\int_0^{33\frac{3}{4}}ydxdy}{\int_0^3\int_0^{21}dxdy+\int_3^9\int_0^{19\frac{1}{8}+\frac{13}{8}x}dxdy+\int_9^{15}\int_0^{33\frac{3}{4}}dxdy}$$

$$=15\tfrac{167}{7884}\text{ miles.}$$

∴ Measure east from beginning $8\frac{163}{876}$ miles, then north $15\frac{167}{7884}$ miles.

[The proposition that, "The point of the area of a triangle, which has the sum of its distances to all other points of the area a *minimum*, is the centre of gravity of the area," which I think holds for other figures, practically solves problem 24, No. 2. I have made out the proof for the triangle but it occupies two pages. R. J. ADCOCK, Larchland, Illinois.]

Also solved by *P. S. BERG.*

A CORRECTION.—On page 246 of the MONTHLY, my remarks in the lower four lines *above* the *Note*, are not true and should be expunged. They were hastily made upon insufficient investigation. In prob. 20, those remarks hold almost true, but in the general problem they can not ever be true. My solution is not at all affected by those misstatements. The solution may be more easily understood by adding, that, "when Sirius rises, some point of the ecliptic is then rising, and as the Sun is always on the ecliptic the Sun must be at that point, in order to rise synchronously with Sirius.

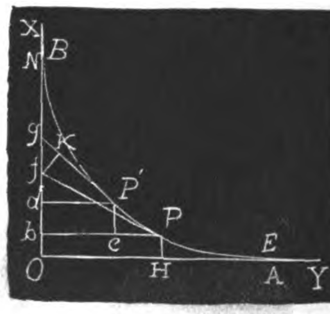
S. H. WRIGHT.

34. Proposed by GEORGE LILLEY, Ph. D., LL. D., Park School, Portland, Oregon.

A hare is at O, and a hound at E, 40 rods east of O. They start at the same instant each running with uniform velocity. The hare runs north, The hound runs directly towards the hare and overtakes it at N, 320 rods from O. How far did the hound run?

III. Solution by J. M. BANDY, A. M., Professor of Mathematics in Trinity High School Trinity, North Carolina.

Let $OE=a$, $ON=b$, $r=$rate of hound, $r'=$rate of hare, $\frac{r}{r'}=n$, ratio of their speed; co-ordinates of $P=(x, y)$, and $z=EP$. By the theory of curves, $dz=\sqrt{1+\frac{dx^2}{dy^2}}\,dy \ldots (1)$. Now $bf=\frac{ydx}{-dy}$, (y being a decreasing function of x is negative).



$\therefore\ Of=x+\frac{ydx}{-dy}$. By the problem, $\frac{EP}{x+\frac{ydx}{-dy}}$ $=\frac{r}{r'}=n$, or $EP=n\left(x+\frac{ydx}{-dy}\right) \ldots\ldots (2)$.

$\therefore\ x+\frac{ydx}{-dy}=\frac{1}{n}\int\sqrt{1+\frac{dx^2}{dy^2}}\,dy \ldots\ldots (3)$. Put $\frac{dx}{dy}=p$, then $dx=pdy$, and we have, $x-py=\frac{1}{n}\int\sqrt{1+p^2}\ dy \ldots (4)$. Differentiating (4) and observing that $dx=pdy$, we have, $-ydp=\frac{1}{n}\sqrt{1+p^2}\,dy \ldots (5)$.

From (5), $\frac{dp}{\sqrt{1+p^2}}=-\frac{dy}{ny} \ldots (6)$. Integrating (6), we have, $\log[p+\sqrt{1+p^2}$ $=-\frac{1}{n}\log y+C=\frac{1}{n}\log\frac{1}{y}+C \ldots (7)$.

When $x=0$, $p=0$, and $y=a$, and $C=\frac{1}{n}\log a$.

$\therefore\ \log[p+\sqrt{1+p^2}=\frac{1}{n}\log\frac{a}{y} \ldots (8)$. Passing from log, $p+\sqrt{1+p^2}$ $=\left(\frac{a}{y}\right)^{\frac{1}{n}} \ldots (9)$. Solving for p, we have, $2p=\left(\frac{a}{y}\right)^{\frac{1}{n}}-\left(\frac{y}{a}\right)^{\frac{1}{n}} \ldots (10)$. But $p=\frac{dx}{dy}$.

$\therefore\ 2dx=\left(\frac{a}{y}\right)^{\frac{1}{n}}dy-\left(\frac{y}{a}\right)^{\frac{1}{n}}dy \ldots (11)$. Integrating (11), $2x=\frac{a^{\frac{1}{n}}y^{1-\frac{1}{n}}}{1-\frac{1}{n}}$ $-\frac{y^{1+\frac{1}{n}}}{a^{\frac{1}{n}}\left(1+\frac{1}{n}\right)}+C \ldots (12)$. When $x=0$, $y=a$, and (12) gives val. of $C=\frac{2na}{n^2-1}$.

$$\therefore\ 2x=\frac{a^{\frac{1}{n}}y^{1-\frac{1}{n}}}{1-\frac{1}{n}}-\frac{y^{1+\frac{1}{n}}}{a^{\frac{1}{n}}\left(1+\frac{1}{n}\right)}+\frac{2na}{n^2-1} \ldots (13.$$

This is the equation of the curve described by the dog, and it is called the "Curve of pursuit." When the dog overtakes the hare, $y=0$, $x=ON=b$. $\therefore$ (12) becomes, $\frac{na}{n^2-1}=b\ldots.(13)$. Solving (13) as a quadratic in n, we have, $n=\frac{1}{2b}(a\pm\sqrt{4b^2+a^2})$, or $n=1.0644+$. Substituting this value of n in (2), remembering that when the dog overtakes the hare, $EP=s$, $x=320$, $y=0$, we have, $s=1.0644\times320=340.624+$ rods.

PROBLEMS.

33. Proposed by Professor ALEXANDER ROSS, C. E., Sebastopol, California.

From a point P without a rectangular field ABC, the distances PA, PB, and PC measured to the corners are, respectively, 70, 40, and 60 chains. What is the area of the field?

34. Proposed by THOS. U. TAYLOR, C. E., M. C. E., Department of Engineering, University of Texas, Austin, Texas.

Given a variable parrallelogram $ABCP$, where P remains fixed. A moves on an irregular plane curve (closed) and C moves on another plane curve (closed) whose plane is parallel to the plane of (A) curve. The generator PC moves completely around and returns to its initial position, AB always moving parallel to PC, and, of course, returns to its initial position. If distances between planes (A) and ($C=h$), show by elementary mathematics and without using theorem of Koppe that volume of solid generated by variable parallelogram $ABCP=\frac{1}{2}h$ (area generated by $AP+$ area generated by BC).

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SPACE.

Space is an entity, outside of the human mind, extended in three directions at right angles to each other, continues, immaterial, immmovable, inflexible and illimitable. It is an entity, sui generis, neither psychical nor physical.

It is cognized but not created by the mind of man and is, doubtless, what it is cognized to be.

A fourth dimension has never been discovered.

Arthur Willink in "The World of the Unseen," pages 90 and 91, locates his hypothetical "Higher Space" in an unknowu direction from our space.

A straight line drawn from us can never reach it and a straight line drawn from the "Higher Space" "towards our space will only pass through one point in our space." For, says he, "If it could pass through more than one point we should know its direction, since two points in a straight line are sufficient to determine the direction of that line."

Principal Campbell in his Philosophy of Rhetoric distinguishes between the "*unintelligible*" and the "*absurd*". Arthur Willink's speculations respecting "Higher Space" are magnificent specimens of the "absurd."

JOHN N. LYLE.

THOSE ASTRONOMICAL CRITICISMS.

To DR. S. H. WRIGHT's criticisms on page 246 of the July-August MONTHLY, I beg to make a few remarks.

I. Without wasting any time and energy in the *discussion* of supplementary hour-angles, *my* method of calculation by running the solar system on Dog-Star time, "gets there" far more expeditiously than does Dr. Wright's *admirable* though elaborate and rather unintelligible method.

II. Since I am sixty miles from my copy of *Bartlett's Spherical Astronomy* from which I substantially *borrowed* the sentence relative to the "reversed crescent" illuminated on the "Waning Moon," I am unable to determine whether I *correctly transcribed this sentence* which Dr. Wright so vigorously attacks; but from what I glean from the *Encyclopedia Britannica* and from the astronomical works of Professors Newcomb, Young, etc., I am led to exclaim: "*Our flag is still there.*"

F. P. MATZ.

LAMBERT'S REASON FOR HOLDING THAT THE PARALLEL-AXIOM NEEDS PROOF.

Lambert called by Kant "der unvergleickliche man" is reported as saying that—"the parallel axiom needs a proof, since it does not hold for the geometry on a sphere."

He here assumes without proof that the parallel-axiom does not hold for the geometry on a sphere. That is, he regards this assumption as axiomatic.

Grant that the parallel-axiom does not hold for *spherical* geometry, what has that to do with its holding or not holding for *plane* geometry?

A *plane* surface and a *spherical* surface have some features in common but others in which they fundamentally differ.

Some things are doubtless axiomatically true for a spherical surface that are not for a plane surface, and *vice versa*.

Some things, also, are demonstrably true for a spherical surface that can be demonstrated not to be true for plane surfaces, and *vice versa*.

Three distinct views emerge in the answers given to the question—Is the parallel-axiom as restricted to a plane surface self evident or does it need proof?

1. Euclid assumes its axiomatic character.

2. Many geometers believe it to be a sound geometrical statement but regard it as needing proof.

3. The Non-Euclideans do not view it either as axiomatic or as demonstrable. They deserve no credit for taking this ground, since they can find no foothold except by doubting or denying that it is either self evident or capable of being proved. Their position, however, is plainly that of geometrical *agnosticism*. To call it geometrical *science* would be a misnomer.

Finally, let us go deeper and ask the question—Is the parallel-axiom considered as a geometrical statement true or false?

The Euclideans respond that it is a true geometrical statement. But if true the statement that cantradicts it must be false. Two propositions that mutually contradict each other can not both of them be true while the laws of non-contradiction and excluded-middle stand unrepealed among the statutes of logic.

The Non-Euclideans answer that they do not know whether the twelfth axiom of Euclid considered as a geometrical statement is true or false.

The geometer who refuses to confess with proud humility that he is in the same exalted condition of learned ignorance respecting geometrical fact must submit to being classed with that large majority who know some things and, also, know that they know them. The Non-Euclidians must not become discouraged, however, if they find the school houses full of geometers incorrigibly persistent in maintaining the *Hypothesis anguli recti*, and an angle sum strictly equal to two right angles.

Lobatschewsky in his theorem 19 proved that the sum of the three angles of a rectillineal triangle can not be greater than two right angles.

It is further believed that in his theorem 23 he could have demonstrated tuat the angle sum can not be less than two right angles, if he had not overlooked the important fact that the sum of two of the angles in each of the triangles constructed and admitted into the series is equal to one right angle + the acute angle, a, common to all the triangles. JOHN N. LYLE.

EDITORIALS.

"THE MONTHLY is a tonic, and an excellent one."—[E. L. Sherwood, A. M., Mississippi Normal College, Houston, Miss.]

PROF. C. A. WALDO, formerly of Greencastle, Ind., is now at the head of the Mathematical Department in Purdue University, Lafayette, Ind.

PROF. P. H. PHILBRICK is now located at Pineville, La., where he is very busily engaged in work for the Kansas City, Watkins & Gulf R. R.

PROF. O. W. ANTHONY, M. Sc., late of the Missouri Military Academy, Mexico, Mo., is now at New Windsor College, New Windsor, Md.

"I THINK the MONTHLY is doing a good work and hope it will be sustained."—[Geo. A. Osborne, S. B., Prof. of Mathematics in Mass. Inst. of Technology, and author of *Osborne's Diff. and Int. Calculus* (1891)].

"I FIND very many articles in it that are interesting and instructive, and I trust the MONTHLY may have a prosperous career."—[William J. Milne, Ph. D , LL. D., New York State Normal College, Albany, N. Y.]

PROF. E. S. LOOMIS, of Cleveland, Ohio, writes, "I find my new field of labor very agreeable, and I am now free to *think* as truth leads me out. I am so constituted that I *can not* 'build to *fit* an idea, but to *find* one'."

THE American Mathematical Society held its Second Summer Meeting at Springfield, Mass., Aug. 27th and 28th, at which time and place were held also meetings of the American Association for the Advancement of Science and several other scientific societies of a national character.

WE have been obliged to issue another double number, from causes which we could not control. We hope our readers will bear with us patiently until we have completed our arrangements for the publication of the MONTHLY for next year. After that time we hope to have the MONTHLY appear regularly each month.

OUR contributors should please observe the following in reference to their contributions: (1) Write out their solutions of problems on substantial paper having a width of from 8 to 10 inches; (2) observe punctuation and capitalization; (3) write each solution on a separate slip of paper; and (4) sign your name to each solution or contribution.

"I PRIZE the MONTHLY very highly indeed. Taken all in all it is the best mathematical journal that has appeared in our country. I only wish I could find time to work out some of its excellent problems, and add a word now and then to its interesting discussions."—[Edward Brooks, A. M., Ph D., Superintendant's Office, 713 Filbert St., Philidelphia, Pa.

UNDER "*Queries & Information*," we have published two brief articles from the pen of Dr. John N. Lyle. These two articles speak for themselves. Dr. Lyle may be regarded as the greatest Anti-Non-Euclidean Geometer in America, and he has furnished many papers for publication in the MONTHLY. These we shall publish as our space permits. *Truth* has nothing to fear at the hands of any one, and if the Non-Euclidean doctrine is true, Dr. Lyle's papers will only aid in establishing it. Every great advance in science, every great discovery in nature, and every great invention has had its crowd of ridiculers; and Non-Euclidean Geometry is no exception. The Editors of the MONTHLY belong to the Non-Euclidean school of thought, even though the knowledge of that school respecting geometrical facts is an "exalted condition of learned ignorance." A school of thought represented by such men as Cayley, Sylvester, Klein, Gauss, Lambert, Lobachevsky, Halsted, Moore, and a great many others, can not be very far wrong "respecting geometric truth."

BOOKS AND PERIODICALS.

Master and Man. By Lev N. Tolstoi. Translated from the Russian by Yekaterina Alexandroona and Dr. George Bruce Halsted. Volume two of the Neomonic Series. Published at THE NEOMON, 2407 Guadalupee St., Austin, Texas.

The translation of this interesting story by Dr. Halsted, has indebted the whole literary world of the English speaking people to him. The story is very interesting from beginning to end and teaches a very useful lesson. This would have been lost to a great majority of the English people had not Dr. Halsted given them this translation. B. F. F.

The Number-System of Algebra, Treated Theoretically and Historically. By Henry B. Fine, Ph. D., Professor of Mathematics in Princeton College. 8vo. cloth, 132 pp. Price, $1.00 Boston, New York and Chicago: Leach, Shewell & Sanborn.

In this little book, are treated concisely the following subjects:

I. Theoretical. The Positive Integer; Subtraction and the Negative Integer: Division and the Fraction; The Irrational; The Imaginary and Complex Numbers; Graphical Representations of Numbers; The Fundamental Theorem of Algebra; Defined by infinite series; The Exponential and Logarithmic Functions.

II. Historical. The Primitive Numerals; Historic System of Notation: The Fraction; Origin of the Irrationals; Origin of the Negative and the Imaginary: Acceptance of the Negative, the general Irrational, and the Imaginary as Numbers: Recognition of the Purely Symbolic Nature of Algebra.

This little book should be read by every teacher of Algebra and by every student desiring to pursue the Modern Higher Mathematical Analysis. The theoretical part is very instructive and the historical part full of many interesting facts.

B. F. F.

Text-Book on Algebra through Quadratic Equations. By Joseph V. Collins, Ph. D., Professor of Mathematics in Miami University. 8vo. cloth, 468 pp.+18 pp. of *Ans.* Price, $1.00. Chicago: Albert, Scott & Co.

This book is supplied with well chosen problems and each subject is presented by numerous illustrative problems. This is a very commendable feature.

The definitions and the explanations are stated with sunlight clearness. On pages 362-366 is a discussion of the Validity of Processes of Solution of Quadratic Equations. This discussion is very appropriate and should be incorporated in all algebras. On pages 441-444 is given a correct demonstration of the theorem of Undetermined Coefficients. The demonstration of this theorem in many of our text-books on algebra is fallacious. We consider the book a credit to the subject of Elementary Algebra. B. F. F.

A Geometrical Treatment of Curves which are Isogonal Conjugate to a Straight Line with respect to a triangle. By Dr. I. J. Schwatt, University of Pennsylvania. Price, $1.00. Boston, New York, and Chicago: Leach, Shewell, and Sanborn.

The first part of the book continues the properties of Steiner's Ellipse and Kiepert's Equilateral Hyperbola, treated as isogonal conjugate curves. The writer has endeavored to give the propositions of the triangle on which the properties of the curves are founded in a pure geometrical way. The book will prove of interest, even to such readers as are only interested in the beautiful properties of the triangle, without regard to isogonal curves. The book cannot be better recommended as it is by the eminent French athematician Vigarie, one of the greatest authorities on the Geometry of theTriangle:--

"The work has been admirably conceived, and in my belief it is the first essay of the kind that has ever been published. I do not doubt that the book will be read with the greatest interest by all those who love and cultivate the geometry of the triangle. I am convinced that the second part of the very interesting memoir will meet with the same success."—VIGARIE.

The second part will contain The Ellipse (*continued*), The Parabola, and Higher Plane Curves. B. F. F.

Geometry Tablet for Written Exercises. For use with any text-book. By Wooster Woodruff Beman, Professor of Mathematics in the University of Michigan, and David Ugene Smith, Professor of Mathematics in the Michigan State Normal School. Boston and Chicago: Ginn & Co.

This tablet is prepared to assist the student in expressing his demonstrations of Original Propositions in a neat and attractive manner. Nearly all of our modern text-books on geometry are well supplied with original propositions, some books having as many as 700 originals.

In geometry classes where a year is devoted to the study, the live and progressive teacher of geometry has all of these propositions worked into the daily exercise; for geometry is now seldom taught as it was twenty years ago. A student is required to do more than merely commit to memory the demonstration of a proposition. He is now required to make the argument of the demonstration his own and until he has acquired the habit to think and to not memorize in the study of geometry, his work is that of a parrot.

The use of this tablet will enable the teacher to make careful criticisms, and the student in turn has an opportunity to study the criticisms and profit by them. I like the tablet so well that I have recommended it to my class. The tablet could be improved by making two perforations at the top so that the leaves could be bound together. B. F. F.

The Basis. A weekly journal devoted to Good Citizenship, Public Peace, Personal Security, etc. Edited by Albion W. Tourgee, Mayville, N. Y. Price, $1.50 per year. Single copy, 5c.

Judge Albion W. Tourgee's Weekly Magazine devoted to Good Citizenship as the only means by which Good Government can be surely attained, has a spicy department of Good Government Clubs, conducted by Rev. Thomas R. Slicer, of Buffalo, a department of our Women Citizens, conducted by Miss Ada C. Sweet, of Chicago, and some bright interesting pages twice a month on Boy and Girl Citizens, by Prof. W. K. Wickes, of Syracuse, besides the strong and original work of the editor in each number.

The Basis is the only journal in the world devoted to citizenship and actual government by the people. It is a thirty-two page weekly, magazine size, independent, outspoken and progressive on all subjects. Though it adheres to the principles of the National Republican party, it does not hesitate to discuss all themes with the freedom from party-bias which characterizes the works of its distinguished editor. It should be in the hands of everyone who desires to see healthful progress. B. F. F.

On The Inscription of Regular Polygons.—This is a Reprint from "The Annals of Mathematics" of an article by our valued contributor Leonard E. Dickson, M. A., Fellow in Mathematics in the University of Chicago. Those who read the author's series of articles in late numbers of the MONTHLY, treating this geometric subject without the use of the customary complex imaginary, will find special interest in the above paper, in which the author avoids the use of trigonometry also, basing his treatment solely upon algebraic and geometric principles. J. M. C.

The Educational News. A weekly Journal of Education. Edited by Dr. Albert N. Raub. Price, $1.00 per year. Published by the Educational News Co., Philadelphia, Pennsylvania.

This active and public spirited journal should be in the hands of every progressive teacher in the land. It will prevent any teacher from getting into ruts—a very common occurrence with certain classes. B. F. F.

Note on Infinite Determinants.—A dissertation presented to the faculty of the University of Mississippi for the degree of doctor of philosophy by Eugene Roberts, of Oxford, Miss. The article is based to a great extent upon von Koch's second paper (Acta Mathematica, t. 16), but in the method of treating the subject, that of considering an infinite determinant as an infinite series, the terms of which are infinite products, the author believes to be entirely new. As the result of special study of the subject Mr. Roberts has produced a very creditable paper. We are indebted to Dr. Hume, of the University of Mississippi, for a copy of the above paper in pamphlet form. J. M. C.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Albert Shaw. Price, $2.50 per year. Single number, 25 cents. The Review of Reviews Co., New York City.

The special features of the October *Review of Reviews* are: Religious Jaurnalism and Journalists, by George P. Morris; the Carnegie Libraries.—Notes on a popular educational movement in the "Greater Pittsburg", by William B. Shaw; Malabeleland under the British South Africa Company, by Sir Frederick Frankland, Bart.: The Maori.—Politics and Social Life of the Native New Zelander, by Lois Becke & J. D. Fitzgerald; the Civil Service Problem in Australia, by the Sec'y of the New South Wales Royal Commission on Civil Service; the Manitoba School Question, by the Attorney-General of the Province. These articles contain numerous illustrations.

B. F. F.

The *American Journal of Mathematics* for October contains the following papers:—"On the Deformation of Thin Elastic Wires", by A. B. Basset; "Investigations in the Lunar Theory", by Ernest W. Brown; "Ueber den Sinn der Windung in den singularen Puncten einer Raumcurve", by Von Otto Staude.

Some Considerations showing the Importance of Mathematical Study. By I. J. Schwatt, Ph. D., University of Pennsylvania. This paper contains the opening address of the Mathematical Department of the third summer meeting of the American Society for the Extension of University Teaching. We are grateful to Dr. Schwatt for a copy of this inspiriting address, which is rich in illustration and forcible in presentation. J. M. C.

The Cosmopolitan. An International Illustrated Monthly Magazine.

Edited by John Brisben Walker. Price, $1.00 per year. Single Number, 10 cents.

The principal articles in the October Number of the *Cosmopolitan* are: Cuba's Struggle for Freedom, by J. Frank Clark; the Greatness of Man, by Richard Le Gallienne; State Universities, by Richard T. Ely, Ph. D., LL. D.; Mowgli Leaves the Jungle Forever, by Rudyard Kipling; Are We Old Fogies? by J. C. Ayres, Capt. U. S. A.

The Jungle Stories which have created so much interest during the spring and summer, end with this number. B. F. F.

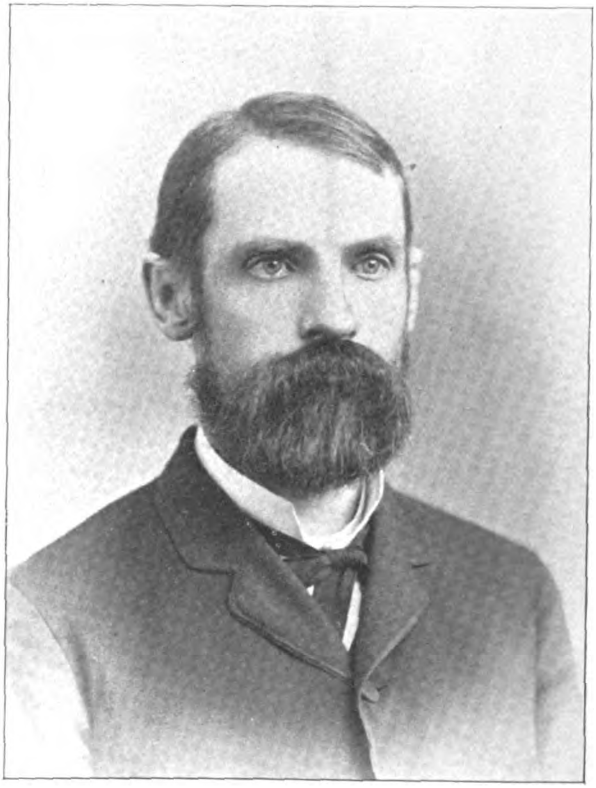
Philosophy of the "New Law in Geometry", leading to the solution of unsolved problems. By Theodore Faber. We extend our thanks to Editor S. C. Gould for a copy of this reprint from *Notes and Queries.*

The following periodicals have been received:—The *Kansas University Quarterly*, (July); The *Monist*, (October); The *Mathematical Gazette*, (May); *Journal de Mathematiques Elementaries*, (July); *Educational Times*, (October); *L'Intermediaire des Mathematiciens*, (September); *Bulletin of the American Mathematical Society*, (July); *El Progresso Matematico*, (June); *Miscellaneous Notes and Queries*, (November).

ERRATA.

JULY-AUGUST NUMBER.

p. 247. ninth line from bottom of page, for "solved" read proved.
p. 248, second line from top of page, for "psendo" read pseudo.
p. " . seventh line from top of page, for "Bell" read Ball. and
p. " . seventh line from top of page, for "Brittanica" read Britannica.
p. " . seventh line from bottom of page, for "Bered" read Berea.
p. 249. eleventh line from bottom of page, for "adopted read adapted.
p. " , fourth line from bottom of page, for "adopt" read adapt.



ORMOND STONE.

THE

AMERICAN MATHEMATICAL MONTHLY.

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BIOGRAPHY.

ORMOND STONE.

BY F. P. MATZ, SC. D., PH. D., PROFESSOR OF MATHEMATICS AND ASTRONOMY IN IRVING COLLEGE, MECHANICSBURG, PENNSYLVANIA.

ORMOND STONE was born January 11, 1847, at Pekin, Tazewell County, Illinois, and was the oldest son of Rev. E. Stone, a travelling preacher of the Illinois Conference of the Methodist Episcopal Church. His father was of New England origin; his mother, Scotch-Irish.

In those early days, Illinois was on the frontier; and the Methodist preachers had large circuits which they changed every year or two. Among the places in which he lived were Canton, Nauvoo, and Carthage. From the last of which places the family moved in 1853 to Cook County, and have remained since then in Northern Illinois.

The boy showed a love for mathematics, when a mere child. At the age of seven, while living in the village of Libertyville, he discovered a copy of a new arithmetic, Adams's, which an attempt had been made to conceal from him. This he read twice, working all the problems each time, in a space of less than six weeks.

The next year his father moved to DeKalb Center, where his interest in mathematics was further advanced by an acquaintance with Dr. Matteson, a notice of whom has already appeared in this journal. A few years afterwards his father was stationed in Chicago, where the youth passed through the public schools of the city.

While still in the High School, the Dearborn Observatory was founded in connection with the old University of Chicago, whither Professor Safford was called and remained in charge until the great fire in 1872. Young Stone soon made his acquaintance and almost immediately became his pupil, and thus began his career as an astronomer.

After graduating at the High School, he taught one year at Racine College; after which he returned to Chicago to continue his studies at the University. In 1869, in company with Professor Safford, he went to Des Moines, Iowa, to observe the great eclipse of that year. While there, he made the acquaintance of the astronomers sent from the Washington Observatory; and as a result, the next spring, he became an *assistant* in that institution. He was assigned to the Meridian Circle, on which he was employed for the next five years.

In 1875, he was called to the *Directorship* of the Cincinnati Observatory. Here, in connection with his assistants, he employed the 11-inch Equatorial of that institution in an extended and practically complete series of measures of the then known southern double-stars north of 30° south declination. Here, also, he commenced his work as *a trainer of young astronomers*, of whom now probably a larger number occupy important astronomical positions than the pupils of any other teacher in America.

In 1882, he was invited *to take charge of the new Leander McCormick Observatory of the University of Virginia.* This had not then been built. The great 26-inch telescope was finally ready for use in the spring of 1885. This building is memorable as possessing the first large dome made by Warner and Swasey. For the *first* time, also, in this country, electricity was applied to the illumination of the circles and micrometer of the great Refractor.

As the southern *double-stars* had been observed at Cincinnati, it was appropriate that he should devote this larger instrument to observations of southern *nebulæ*. As a result, hundreds of new nebulæ were discovered; and in 1893 there was published a catalogue of the micrometric measurements of the positions of southern nebulæ,—the only extended series of such measurements ever made in this country.

Meanwhile Professor Stone has made a special study of the great nebula of Orion, including a great number of photometric observations of the *condensations* of the Huyghenian region, and of the stars, especially of the *variables*, contained therein.

On the completion of the tenth volume of the *Analyst*, published by the late Dr. Hendricks, of Des Moines, Iowa, when that journal ceased to exist, Professor Stone began the publication of the *Annals of Mathematics*. For a time the editorship was shared with him by his colleague, Professor William M. Thornton; but at the close of the second volume, Professor Stone took entire charge, and the journal has been in his hands ever since.

In this elegantly printed bi-monthly journal, some very select problems are proposed for solution; and the solutions of the problems proposed are published as soon as possible. The *main object* of the publication of the *Annals of Mathematics*, by Professor Stone, is *to encourage mathematical research.*

Professor Stone is a brother of Mr. Melville E. Stone, of Chicago, the well-known journalist, the founder of the Chicago *Daily News*, and the general manager of the Associated Press.

Professor Stone has written various papers on mathematical and astronomical subjects, which have appeared from time to time in the *Astronomische Nachrichten*, in *Gould's Astronomical Journal*, and in the *Annals of Mathematics.*

Professor Stone is also a member of a number of learned societies. In 1888 he was Chairman of the section of Mathematics and Astronomy of the American Association for the Advancement of Science; and he is at present a member of the Council of the American Mathematical Society.

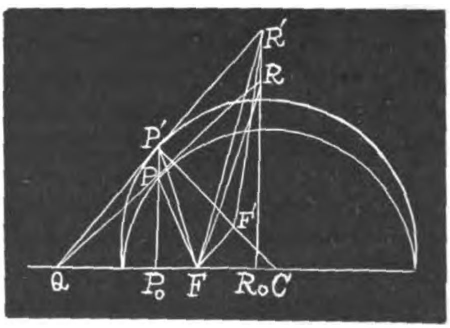
AN ELEMENTARY DERIVATION OF THE LAW OF GRAVITATION AS APPLIED TO PLANETARY MOTIONS.

By ORMOND STONE, University of Virginia.

The following derivation of the law of gravitation from Kepler's first two laws of planetary motion without the use of the machinery of the infinitesimal calculus is a modification of that given by Moebius. The loss by fire of a large portion of the library of the University of Virginia prevents my giving the place in his works where it may be found. As given by Moebius a slight knowledge of solid geometry is required; as here given all the operations are performed in the plane of the orbit. The mass of the planet has been neglected.

Draw a circle having the major axis of the orbit as a diameter. Assume a point P' having such a motion that it is always at the intersection of the circumference of this circle and a straight line drawn through the planet P perpendicular to the major axis of the planet's orbit. The components of the velocities of P and P' in the direction parallel to the major axis are thus equal.

Draw QR tangent to the ellipse at P, and QR' tangent to the circle at P'. Q is situated on the major axis extended. If $PR = V$ represent the velocity of P and $P'R' = V'$ represent the velocity of P', RR' will be parallel to PP'. Let P_0 and R_0 be the intersections of PP' and RR' with the major axis of the orbit. Then by a property of the ellipse



$$P_0P = P_0P' \cos \varphi,\ R_0R = R_0R' \cos \varphi,$$

in which φ is the angle whose sine is e, the eccentricity of the orbit.

By one of Kepler's laws the sun is at F, the focus of the ellipse. PRF represents the areal velocity of P, and $P'R'F$ the areal velocity of P' with reference to F. As is easily seen,

$$PRF = P'R'F \cos \varphi ;$$

whence, since by one of Kepler's laws PRF is constant, $P'R'F = c'$ is also a constant, and the acceleration of P' is directed toward F (see Young's General Astronomy, Art. 406).

Let A and A' be the total accelerations of P and P', and A_0 and A_0' be the components of these accelerations parallel to the major axis of the ellipse. Evidently

$$\frac{A_0}{A} = \frac{P_0F}{PF}, \qquad \frac{A_0'}{A'} = \frac{P_0F}{P'F};$$

whence, since $A_0 = A_0'$,

$$\frac{A}{A'} = \frac{P'F}{PF}. \qquad (1)$$

Let C be the center of the ellipse, and F'' the foot of the perpendicular from F on $P'C$. The component of A' in the direction $P'C$ is

$$A' \cos FP'F'' = A' \frac{F''P'}{FP'} = \frac{V'^2}{a} \qquad (2)$$

(see Young's General Astronomy, Art. 411).

Put $\angle FCP' = E =$ eccentric anomaly, and $FP = r =$ radius vector. We have also

$$CF = ae,$$
$$CF'' = ae \cos E,$$
$$F''P' = CP' - CF'' = a(1 - e \cos E),$$
$$P_0P = a \cos \varphi \sin E,$$
$$FP_0 = a\,(\cos E - e).$$

The last two equations give

$$FP = \sqrt{\overline{FP_0}^2 + \overline{P_0P}^2} = a(1 - e \cos E);$$

whence

$$F''P' = FP = r,$$

and (1) and (2) give

$$A = A' \frac{F''P'}{FP'} = \frac{V'^2}{a}. \qquad (3)$$

Since FF' is parallel to $P'R'$, the area of $P'F'R'$ is equal to that of $P'FR'$, which has already been shown to be constant; whence

$$F'P' \times P'R' = rV' = 2c', \text{ or } V' = \frac{2c'}{r}.$$

Substituting this in (3), we have

$$A = \frac{4c'^2}{a} \cdot \frac{1}{r^2}. \qquad \text{Q. E. D.}$$

A NOTE ON MEAN VALUES.

By E. H. MOORE, Ph. D., Professor of Mathematics in the University of Chicago.

A problem in averages or mean values usually reads thus:

(A) *Given a certain totality* $\Omega[\psi]$ *of objects* ψ, *and a certain function* $f(\psi)$ *of every object* ψ; *required the mean value* f_Ω *of the* $f(\psi)$ *for the* ψ*'s of the totality* $\Omega[\psi]$.

If the totality $\Omega[\psi]$ contains a *finite* number n of objects $\psi - \psi_1, \psi_2 \ldots \psi_n$ — then we have the formula

$$(1) \qquad f_\Omega = \frac{\sum_{i=1}^{n} f(\psi_i)}{n}.$$

If the totality $\Omega[\psi]$ does not contain a finite number of objects, then the problem as stated (A) is *indefinite*. [The solution (1) cannot be directly generalized. To say that the number n is ∞ means merely that the totality $\Omega[\psi]$ is without number, that there is *no* such *number n*.]

To make (A) *definite we must supplement it by an explicit statement of a law of distribution of the objects* ψ *over the totality* $\Omega[\psi]$. In the ordinary cases this law of distribution makes ψ depend uniquely upon certain m independent variables $u_1, u_2 \ldots u_m$, write it $\psi = \psi(u_1 \ldots u_m)$, in such a way that the totality $\Omega[\psi]$ defines a certain totality $\bar{\Omega}[u_1 \ldots u_m]$, and the function $f(\psi)$ becomes $f(\psi) = \bar{f}(u_1 \ldots u_m)$. Now if the m-ple definite integrals

$$(2) \qquad I_1 = \int \ldots \int \bar{f}(u_1 \ldots u_m)\, du_1 \ldots du_m, \quad I_2 = \int \ldots \int du_1 \ldots du_m$$

taken over the totality $\bar{\Omega}[u_1 \ldots u_m]$ have definite meaning, (whether *or not* the integral can be explicitly evaluated in terms of the commoner functions) then as the mean f_Ω of $f(\psi)$ for this distribution of ψ over $\Omega[\psi]$ we have

(3) $$f_\Omega=\frac{I_1}{I_2}.$$

As has appeared time after time in all periodicals having a department devoted to mean values, when a problem is proposed in the indefinite form (A), different solvers introduce different laws of distribution, [each one the law appearing to him the most natural, or often enough, the law for which he can explicitly evaluate the integrals (2)]. These laws of distribution should be introduced explicitly as needed to make the problem definite. The possible laws of distribution are without number.

There is *no such thing* as THE *correct* solution of a problem stated in the indefinite form (A).

I make these quite obvious remarks for the sake of those persons who enjoy plunging into the whirl of the integrations without due meditation on the essential nature of the problem they are attacking.

The University of Chicago, November 5, 1895.

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from September-October Number.]

SYSTEMS OF INTRANSITIVITY 3, 3.

(a) By multiplying every substitution of one group by every substitution of the other group we obtain

1. A group of order 36, viz:

1	1*
abc	def
acb	dfe
ab	de
ac	df
bc	ef

*This notation indicates that every substitution on one side of the line is multiplied into every one on the other side.

2. A group of order 18, viz:

$$\begin{array}{r|l} 1 & 1 \\ abc & def \\ acb & dfe \\ & de \\ & df \\ & ef \end{array}$$

3. A group of order 9, viz:

$$\begin{array}{r|l} 1 & 1 \\ abc & def \\ acb & dfe \end{array}$$

(*b*) By making the identical substitutions in the two systems correspond we obtain

4. A group of order 6, viz:

$$\begin{array}{c} 1 \\ abc.def \\ acb.dfe \\ ab.de \\ ac.df \\ bc.ef \end{array}$$

5. A group of order 3, viz:

$$\begin{array}{c} 1 \\ abc.def \\ acb.dfe \end{array}$$

(*c*) By multiplying the divisions according to a self-conjugate subgroup of one group into the corresponding divisions of the other we obtain

6. A group of order 18, viz:

$$\begin{array}{r|l} 1 & 1\,* \\ abc & def \\ acb & dfe \\ \hline ab & de \\ ac & df \\ bc & ef \end{array}$$

*This notation implies that the divisions of one column are to be multiplied into the corresponding divisions of the other column. Thus, *ab* is multiplied into *de*, *df*, and *ef*.

Systems of Intransitivity 2, 4.

There are five transitive groups of degree four, viz :

$$(abcd)all,\ (abcd)pos,\ (abcd)_8,\ (abcd)_4,\ (abcd)cyc$$

$$(abcd)_8 = 1 \ \ ac \ \ ab.cd \ \ abcd$$
$$bd \ \ ac.bd \ \ adcb$$
$$ad.bc$$

$$(abcd)_4 = 1 \ \ ab.cd \qquad (abcd)cyc = 1 \ \ ac.bd \ \ abcd$$
$$ac.bd \qquad\qquad adcb$$
$$ad.bc$$

By multiplying every substitution in each of these groups by $1,ef$ we clearly obtain five additional intransitive groups of degree six whose orders are 48, 24, 16, 8, and 8 respectively. It remains to find the groups which can be obtained by multiplying the self conjugate subgroups which include half the substitutions of these groups by 1 and the remaining substitutions by ef.

Since $(abcd)all$ and $(abcd)cyc$ have one such self conjugate subgroup apiece, they give rise to two groups of the required type. $(abcd)_8$ and $(abcd)_4$ have three such subgroups apiece, but in the latter case they are all conjugate, as we have already seen ; hence we obtain the following four groups from these two groups :

(1)		(2)		(3)		(4)	
1		1		1			
ac		$ab.cd$		$ac.bd$			
bd	1	$ac.bd$	1	$abcd$	1	1	
$ac.bd$		$ad.bc$		$adcb$		$ab.cd$	1
$ab.cd$		ac		ac		$ac.bd$	ef
$ad.bc$	ef	bd	ef	bd	ef	$ad.bc$	
$abcd$		$abcd$		$ab.cd$			
$adcb$		$adcb$		$ad.bc$			

We have thus found that there are eleven intransitive groups of degree six whose systems of intransitivity are 2 and 4.

Systems of Intransitivity 2, 2, 2.

If we restrict ourselves to the first two systems we obtain an intransitive group of degree four. Our problem is thus reduced to the finding all the intransitive groups that can be obtained by combining one of the two intransitive groups of degree four with the group $1,ef$.

By multiplying every substitution of these two groups into $1,ef$ we obtain two groups whose degrees are four and eight respectively. By multiplying one into the self conjugate subgroups of these groups and *ef* into the remaining substitutions we obtain the following additional groups :

$$\begin{array}{c|c} 1 & 1 \\ \hline ac.bd & ef \end{array} \qquad \text{and} \qquad \begin{array}{r|l} 1 & \\ ac.bd & 1 \\ \hline ac & ef \\ bd & \end{array}$$

We have now worked over the entire field and found twenty-one intransitive groups of degree six. These may be written as follows :*

LIST OF INTRANSITIVE GROUPS OF DEGREE SIX.

Order	No.	Group
2	1	$(ac.bd.ef)$
3	1	$(abc.def)$cyc
4	1	$(ab.cd)(ef)$
	2	$\{(ab)(cd)(ef)\}$pos
	3	$\{(abcd)$cyc$(ef)\}$pos
	4	$\{(abcd)_4(ef)\}$dim
6	1	$(abc.def)$all
8	1	$(ab)(cd)(ef)$
	2	$(abcd)$cyc(ef)
	3	$(abcd)_4(ef)$
	4—6	$\{(abcd)_8(ef)\}$
9	1	(abc)cyc(def)cyc
16	1	$(abcd)_8(ef)$
18	1	(abc)all(def)cyc
	2	$\{(abc)$all(def)all$\}$pos
24	1	$\{(abcd)$all$(ef)\}$pos
	2	$(abcd)$pos(ef)
36	1	(abc)all(def)all
48	1	$(abcd)$all(ef)

From what has been said it may be seen that intransitive groups of a given degree n can be readily constructed provided the transitive groups degree k are known, where $k=2, 3, 4, \ldots . n-2$. When n does not exceed 9 it is not difficult to write down all the possible intransitive groups, but for larger values of n the number of groups is so large that the construction of all such groups becomes comparatively quite laborious. The transitive groups seem much more important than the intransitive ones and we proceed to give methods by which they may be constructed.

Definition.—If a transitive group is such that any β letters may be replaced

*Cf. Professor Cayley : *Quarterly Journal of Mathematics*, vol. 25, pp. 71—79.

by any required set of β letters taken in any one of the $\beta!$ possible ways it is said to be β-fold transitive. If $\beta=1$ the group is one-fold or *simply* transitive.

Definition.—A transitive group whose letters can be divided into systems such that all the substitutions of the group perform upon these systems only one or both of the operations (1) interchanging the systems as units and (2) permuting the letters in the systems, is called a *non-primitive group.** All other transitive groups are called *primitive.*

We shall consider separately the construction of these two classes of transitive groups.

Construction of the Non-Primitive Groups.

Since the non-primitive group must be transitive it is always possible to replace any system by any required system by means of some substitution of the group. Hence it follows that the same number of letters is found in each system.

If $G=s_1, s_2, \ldots, s_g$ is any non-primitive group and $G_1=s_1, s_2, \ldots, s_{g_1}$ are the substitutions of G which do not interchange any of the systems then will G_1 be an *intransitive self conjugate subgroup of G.*

G_1 is a group because the substitutions

$$s_\alpha s_\beta \quad \begin{array}{l} \alpha=1, 2, \ldots g_1 \\ \beta=1, 2, \ldots {}_2g_1 \end{array}$$

cannot interchange any of the systems. They must therefore be found in G_1 since they certainly belong to G.

G_1 is a self conjugate subgroup of G because

$$s_\gamma^{-1} G_1 s_\gamma \quad g_1 < \gamma < g+1$$

must be a subgroup of G, which does not interchange the systems (the interchange effected by s_γ^{-1} being restored by s_γ) and must therefore be G_1 itself.

We may suppose that the transitive constituents of G_1 form the systems. For if $a_1, a_2, \ldots, a_e$ and $b_1, b_2, \ldots, b_k$ are transitively connected by G_1 then G contains some substitution s_γ which replaces a_1 by b_1. We proved above that

$$s_\gamma^{-1} s_\alpha s_\gamma = s_\beta \quad \begin{array}{l} \alpha=1,2, \ldots, g_1 \\ \beta=1,2, \ldots, g_1 \end{array}$$

Suppose s_α so chosen that it replaces a_1 by a_2. Then it follows that s_γ replaces a_2 by one of the given b's. Similarly, we may show that s_γ replaces all of the given a's by the given b's. Since the a's and b's can be interchanged

*Such a group must be *simply* transitive, for if $\beta>1$ it would be possible to replace two letters of the same system by letters of different systems.

in this course of reasoning we have that $e=k$, or, more generally speaking, that

$$s_\gamma \qquad g<\gamma<g+1$$

either does not interchange any elements of two transitive constituents of G_1 or it interchanges all. Hence the transitive constituents satify the definition of systems.*

We are now prepared to see that the construction of non-primitive groups consists of two operations :

1. The construction of an intransitive group G_1 whose transitive constituents are conjugate groups.†

2. The construction of the substitutions of G which interchange the systems.

The first of these operations is a special case under the construction of intransitive groups and needs therefore no further attention. With respect to the second we shall first consider some special cases and then take up the general problem.

[To be Continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton), Ph. D., (Johns Hopkins), Member of the London Mathematical Society, and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from September-October Number.]

SCHOLION II, *in which is weighed the idea of that brilliant man Giovanni Alfonso Borelli in his Euclides Restitutus.*

This most learned author blames Euclid, because he defines parallel straight lines to be those, *which lying in the same plane do not meet on either side, even if produced into the infinite.*

He offers as ground for his accusation, that such relation is unknown: *first*, he says, *because we are ignorant whether such infinite non-concurrent lines can be found in nature; then also because we cannot perceive the properties of the infinite, and hence a relation of this sort is not clearly cognized.*

But with reverence for so great a man it may be said: can Euclid be blamed, because (to bring forward one among innumerable examples) he de-

*Cf. Netto's Theory of Substitutions (Cole's edition), §67.

†It has not been proved that all groups of this form can be used for G_1 but that every G_1 is of this form. The former cannot be proved.

fines *a square* to be *a figure quadrilateral*, *equilateral*, *rectangular ;* when it may be doubted, whether a figure of this sort has place in nature ? He could, say I, most justly 'have been blamed, if,. before as a Problem demonstrating the construction, he had assumed the aforesaid figure as given.

But that Euclid is free from this fault follows manifestly from this, that he nowhere presumes a square by itself explained, except after Proposition 46 of the First Book, in which in form of a problem he teaches, and demonstrates *the description from a given straight line, of the square as defined by him.*

In the same way therefore Euclid ought not to be blamed, because he defined parallel straight lines in this manner, since he nowhere assumes them as given for the construction of any problem, except after Proposition 31 of book first, in which as a Problem he demonstrates, how *should be drawn from a given point without a given straight line a straight line parallel to this*, and indeed according to the definition of parallels given by him, *so that produced indeed into the infinite on neither side do they meet one another.* And what is more; he demonstrates this without any dependence from the Postulate here controverted. And so Euclid demonstrates without any petitio principii that *there can be found in nature two such straight lines, which* (lying in the same plane) *protracted on each side into the infinite never meet*, and therefore *makes clearly known to us that relation*, by which he defines parallel straight lines.

Let us continue onward, whither the scrupulous accuser of Euclid invites us. Parallel straight lines he calls any two straights *AC*, *BD*, which stand at right angles to one certain straight *AB* (fig. with me 21). I admit, that a definition of this sort is set forth *by a state* (as he says) possible and most evident; since (Eu. I, 11.) from any point in the given straight a perpendicular can be erected.

But precisely this both possibility and clearness I have just now demonstrated about the definition propounded by Euclid.

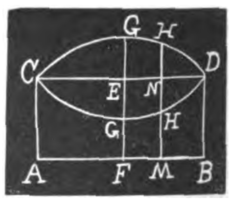


Fig. 21.

Wherefore remains only to compare that known Postulate of Euclid with this the other like postulate, which must be used for farther progress after new definition of parallels.

But behold this other postulate in Clavius (to whom Borelli himself expressly refers) in the Scholion after Proposition 18, book first: If a straight line, as suppose *BD* upon another straight, as suppose *BA*, moves transversely making with it at its extremity *B* always right angles, its other extremity *D* describes a line also straight, until this *BD* shall have come to congruence with the other equal *AC*. I acknowledge the fitness of the postulate, that thence a transit may be made to demonstrating that other Euclidean, upon which certainly at length must be supported all remaining geometry. For Clavius had previously declared; that a line, of which all points are equally distant from a certain assumed straight *AB;* as assuredly is (from the hypothesis of the aforesaid construction) the line *DC;* this line also must be straight; because certainly it will be of this sort, that all its intermediate points *lie ex aequo* (such is the definition of a straight line) between its extreme points D, and C; *lie ex aequo*, say I, since all are equally distant from this assumed straight AB, truly as much as is the length of this BD, or AC. In this place Clavius introduces the example of the circular line, of which we will speak more conveniently below; where I will show the clearest disparity in this part between the straight line and circle.

For meanwhile I say it is not sufficiently evident, whether the line described by this point D is rather the straight DC than a certain curve DGC either convex or concave toward the side of this BA.

For if from the point F bisecting this BA a perpendicular is supposed erected, which meets the straight DC in E, and the aforesaid curves in G, and G, it follows surely (from P. II.) that the angles at the point E will be right; whatever line DC is understood at length as described in this motion by the point D; and moreover (from an easily understood superposition) the angles at the points G will be equal according as the one or the other curve DGC may be described.

But again; any point M in AB being assumed; if a perpendicular is erected, which meets the straight DC in N, and the aforesaid curves in H and H, I will prove a little later that the angles on both sides at the point N will be right, is as far indeed as this straight DC is supposed generated by the point D in that motion of its, or in as far as the straight MN is decided equal to this BD.

But if one or the other curve DHC is supposed generated; from the like aforesaid easy superposition will be demonstated that again the angles MHD, MHC on both sides will be equal, wherever in the one or the other described curve the point H may be assumed, from which to the straight line AB lying under the perpendicular HM is understood as let fall. But of this thing more fully and more scrupulously in the other part of this book, where it has its proper place.

To what end therefore, you will say, this untimely anticipation?

To this end, say I; lest from this indubitable property of the line generated in this manner, proved by me most rigorously in the aforesaid place; and indeed beyond any defect of any sort infinitely small; we may decide precipatately that the line can be only the straight.

Obviously the nature of the straight line must here be investigated more profoundly, without which geometry scarcely grown beyond infancy must there remain. Therefore in this affair cannot be blamed a greater investigation of a certain most exact verity.

Nor yet do I here deny, but that by certain most accurate physical experimentation can be discovered, that the line DC generated by this motion might be determined not other than a straight line.

But in so far as it may be here allowable to cite physical experimentation, I may forthwith bring forward three demonstrations physico-geometric to sanction the Euclidean Postulate.

Therewith I do not speak of physical experimentation extending into the infinite, and therefore impossible for us; such as of course would be required to the cognizing, that all points of the straight join DC are equidistant from the straight AB, which is supposed to be in the some plane with this DC.

For a single individual case will be sufficient for me; as suppose, if, the straight DC being joined, and any one point of it N being assumed, the perpendicular NM let fall to the underlying AB is ascertained to be equal to BD or AC. For then the angles on both sides at the point N would be equal (P. I.) to the angles corresponding to them at the points C and D, which again (from the same P. I.) would be equal *inter se*. Wherefore the angles on both sides at the point N, and therefore also the remaining two will be right.

Therefore we will have a case for the hypothesis of right angle; and consequently (by PP. V. and XIII.) we will have demonstrated the Euclidean Postulate. And this may be the first demonstration physico-geometric.

I pass over to the second. Let there be a semicircle, of which the center is D, and diameter AC. If then (fig. 17) any point B is assumed in its circumference, to which AB, CB joined are ascertained to contain a right angle, this single case will be sufficient (as I have demonstrated in P. XVIII.) for establishing the hypothesis of right angle, and consequently (from the aforesaid P. XIII.) for demonstrating that well known Postulate.

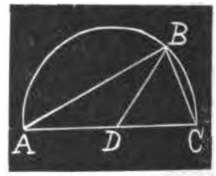


Fig. 17.

There remains the third demonstration physico-geometric, which I think the most efficacious, and most simple of all, inasmuch as it rests upon an accessible, most easy, and most convenient experiment.

For if in a circle, whose center is D, are fitted (fig. 22) three straight lines CB, BL, LA, each equal to the radius DC, and it is ascertained that the join AC goes through the center D, this will be sufficient for demonstrating the assertion.

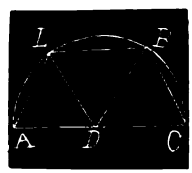


Fig. 22.

For, DB, DL being joined, we will have three triangles, which (from Eu. I. 8 and 5) not only will be equiangular to one another, but also singly for themselves. Therefore since the three angles together at the point D, indeed ADL, LDB, BDC are equal (by Eu. I. 13) to two right angles; also the three angles together of each of these triangles will be equal to two right angles, as suppose of the triangle BDC. Wherefore (from P. XV.) will be established hence the hypothesis of right angle; and consequently (from the already used P. XIII.) that Postulate will be demonstrated.

But if, before all attempt whether at demonstration or at graphic exhibition, one wishes to compare *inter se* those two postulates, I grant indeed the Euclidean may appear more obscure or even liable to objection. But after the graphic exhibition, which I reserve for Scholium IV. following, it will appear *vice versa* that the Euclidean Postulate indeed can retain the dignity and name of postulate, but the other ought rather to be reckoned among the theorems.

But here I must explain (as a little above I have promised I was about to do) the manifest disparity in this relation between the circular line and the straight line. Now the disparity arises from this; that a line is called straight in reference to itself; but is called circular, as suppose (fig. 23) *MDHNM*, not in reference to itself, but to something else, forsooth to a certain other point *A* existing in the same plane with it, which is its center.

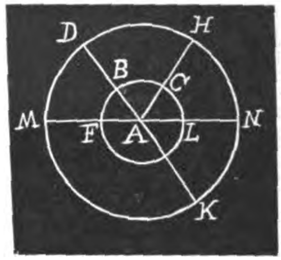


Fig. 23.

The consequence therefore is, as is demonstrated most excellently by Clavius, that the line *FBCL* existing in the same plane with it, and whose points are all equidistant from the aforesaid *MDHNM*, is also itself circular, truly equidistant in all its points from the common center *A*. That in fact *BD*, which is the continuation in a straight of *AB*, is the measure of the distance of that point *B* from this circle *MDHNM* follows from this; because (from Eu. III. 7, which is independent of the postulate here in controversy) this is the smallest of all, which can fall from this point upon this circumference. The same holds of the remaining *CH*, *LN*, *FM*.

Since therefore also the wholes *AM*, *AD*, *AH* are equal as radii from the center *A* to the line assumed circular *MDHNM*; and also the sections *FM*, *BD*, *CH*, *LN* are equal, which obviously are the measure of the equal distance of all points of that line *FBCLF* from this line presumed circular *MDHNM*; the consequence plainly is, that equal likewise are the remainders *AF*, *AB*, *AC*, *AL*, and therefore also this line is a circle with the same center *A*.

But now likewise, for demonstrating, that the line *DC* (fig. 21) generated through such a motion by the point *D* is a straight line will the equidistance of all its points from the underlying straight *BA* be sufficient? In no way.

For a line is called straight absolutely in reference to itself, or in itself, doubtless as *lying ex aequo between its points*, and especially end points, so that these remaining unmoved it cannot be revolved into occupying a new place. Unless this state in some way be demonstrated of this *DC* it will never be certain that this is a straight line, whatever relation finally is supposed or demonstrated of all its points to the underlying straight *AB* in the same plane; but especially we must not say analogically that no other line in this plane will be straight which in all its points is not equidistant from this line *AB* supposed straight.

Nor finally do I wish this dictum of mine so taken, as if I think it cannot be demonstrated, that the line thus generated is itself a straight line, except after truth demonstrated of the controverted postulate; since rather I myself will demonstrate it toward the end of this book, for confirming the like postulate itself.

[To be Continued.]

ASTRONOMICAL DETERMINATION OF THE TIME OF THALES.

By F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

According to Pliny, the *cosmical setting* of the Pleiades, during the time of Thales of Miletus, occurred twenty-five days after the Vernal Equinox.

The geographical latitude of Miletus is $\lambda = +37^\circ\ 30'$. The brightest star of the Pleiades is *η Tauri*, popularly known as *Alcyone*. The apparent place for upper transit of η Tauri, at Washington, for the Mean Solar Date, January 9.3, 1895, is, according to Professor Simon Newcomb, characterized by a *Right Ascension*, $\alpha = 3h.\ 41m.\ 15.49s.$, and by a *Declination*, $\delta = +23^\circ\ 47'\ 0''.4$.

From these data the hour-angle of η Tauri is determined to be $h = 109^\circ\ 45'\ 54''.6, = 7h.\ 19m.\ 3.64s.$ Hence the *sidereal time* of the *rising* of η Tauri is $T_r = 20h.\ 22m.\ 11.85s.$, and *that* of the *setting* is $T_s = 11h.\ 0m.\ 19.13s.$; that is, η Tauri is $T = 14h.\ 38m.\ 7.28s.$ of sidereal time above the horizon.

This problem can now be solved in four different ways; and possibly the most expeditious method of *mathematical* solution, is to make $T_s = (180^\circ - \Theta) = 165^\circ\ 4'\ 46''.95$. Therefore $\Theta = 14^\circ\ 55'\ 13''.05$.

By a calculation analogous to that in our second solution of the "Dog-Star Problem," we find $\frac{1}{2}(Z+V) = 87^\circ\ 27'\ 17''.81$, and $\frac{1}{2}(Z-V) = 68^\circ\ 54'\ 20''.94$. Therefore, $Z = 156^\circ\ 21'\ 38''.75$, and $V = 18^\circ\ 32'\ 56''.87$; and $VZ = 39^\circ\ 57'\ 0''.032$.

The value of the obliquity of the ecliptic, as furnished us by Professor Young, of Princeton, is $\omega = 23^\circ\ 27'\ 18''.82$; and, therefore, $\angle ZV\Sigma = 90^\circ - (V + \omega) = 47^\circ\ 59'\ 44''.31$.

Taking into consideration *Refraction* and *Semidiameter*, we have $Z\Sigma = 90^\circ\ 50'$. Ignoring the method of solution by *right-angled* spherical triangles, we obtain $\angle Z\Sigma V = 28^\circ\ 30'\ 10''.876, = \angle\Sigma$. Therefore, $\frac{1}{2}(V+\Sigma) = 38^\circ\ 14'\ 57''.59$, and $\frac{1}{2}(V-\Sigma) = 9^\circ\ 44'\ 46''.72$; also, $\frac{1}{2}(v+\sigma) = 65^\circ\ 23'\ 30''.02$. Hence $V\Sigma = 120^\circ\ 13'\ 11''.21$, and $(180^\circ - V\Sigma) = 59^\circ\ 46'\ 48''.79, = L$.

With the number of mean days in Hansen's sidereal year taken as the basis, we have for the Sun's mean daily motion $m = 59'.136554, = 59'\ 8''.19324$; and in 25 days, we have $25m = 24^\circ\ 38'\ 24''.828, = M$.

The *Precessional Slip*, therefore, is $\mathbf{P} = (L - M) = 35^\circ\ 8'\ 23''.96, = 126503''.96$.

After adopting Struve's constant of annual precession as given in *Young's General Astronomy*, p. 528, $p = 50''\cdot264 + 0''.000227(t - 1900)$, and then ignoring the right-hand term, we have $\mathbf{T} = \mathbf{P} / p = 2516.79$ years, or 2517 years, to be counted backward (*into the past*) from January 9, 1895, in order to determine the time of Thales.

This count brings us to January 9, 622 B. C. Dividing by the Struvian constant of *mean* annual precession, we have January 9, 628 B. C. Dividing also by the *corrected* Bessellian constant of annual precession, $50''.2479$, we have January 9, 623 B. C.

According to the *Encyclopædia Britannica*, Thales of Miletus lived from 640 B. C. to 546 B. C.; and on May 28, 585 B. C. occurred the total eclipse of the sun, which he had predicted many years before its occurrence.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

52. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

By selling a horse for H=\$150 cash, I gain p=20 per cent. At what price should I sell the horse and wait d=90 days, money worth m=6 per cent., in order to gain q=25 per cent.?

Solution by the PROPOSER.

Obviously the *cost* of the horse is $C=\left[1 \div \left(1+\frac{p}{100}\right)\right]$ of \$$H$,=\$125; and, consequently, the *selling price*, in order to gain q per cent., must be $P'=\left(1+\frac{q}{100}\right)$ of \$$C$,=\$156.25, which must be divided by the *proceeds* of \$1 for d days at m per cent.; that is, the *required* result becomes

$$P=\left(\frac{100+q}{100+p}\right)\left(\frac{36000}{36000-dm}\right) \text{of } \$H, =\$158.62944.$$

Also solved by *H. C. WILKES.*

53. Proposed by P. S. BERG, Larimore, North Dakota.

\$500. WOOSTER, OHIO, September 2nd, 1886.

One year after date we, or either of us, promise to pay to the order of J. M. W. Five Hundred Dollars for value received with interest at 7 per cent. from date.

J. C.
M. C.

Endorsed, May 13, 1893, \$75.00.
" September 1, 1894, \$300.00.
What was due April 1st, 1895?

Solution by H. C. WILKES, Skull Run, Virginia.

As the first payment \$75 is less than the interest then due, compute the interest to the time of the second payment.

Principal, September 2, 1886,		$500.
Interest to September 1, 1894, 8 years less 1 day,		279.90
Interest on Interest = Interest on $35 for 28 years less 8 days,		68.55
Amount of Principal and Interest due September 1, 1894,		$848.45
Payments,	$375.	
Interest on $75 from May 13, 1893, to September 1, 1894,	7.26	
		$382.26
Amount due September 1, 1894,		$466.19
Interest for 7 months,		19.03
Amount due April 1st, 1895,		$485.22

PROBLEMS.

56. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

A, B, and C can walk at the rate of $a=3$, $b=4$, and $c=5$ miles, per hour. They start from Washington, at $m=1$, $n=2$, and $p=3$ o'clock, P. M., respectively. When B overtakes A, he is ordered (by A) back to C. When will B and C meet? Suppose B had ordered A back to C, when would A and C meet? In case all three continue walking ahead, at what time will they meet?

57. Proposed by L. B. FRAKER, Weston, Ohio.

Suppose that in a meadow the grass is of uniform quality and growth and that 6 oxen or 10 colts could eat up 3 acres of the pasture in $\frac{18}{25}$ of the time in which 10 oxen and 6 colts could eat up 8 acres; or that 600 sheep would require $2\frac{6}{7}$-weeks longer than 660 sheep to eat up 9 acres.

It what time could an ox, a colt, and a sheep together eat up an acre of the pasture on the supposition that 589 sheep eat as much in a week as 6 oxen and 11 colts? By Arithmetic, if possible.—Hunter's Arithmetic. (Unsolved in *School Visitor*.)

Solutions of these Problems should be received on or before January 1, 1896.

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

48. Proposed by SETH PRATT, C. E., Assyria, Michigan.

What is the interest of $500 for 10 years at 10 per cent. per annum, when the intervals of compounding are infinitely small?

I. Solution by Professor E. W. MORRELL, Department of Mathematics, Montpelier Seminary, Montpelier, Vermont; and BENJ. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio.

The formula for P dollars at compound interest for n years payable q times a year at rate r, is $P(1+\frac{r}{q})^{qn}$. In this case q is infinity; let $\frac{r}{q}=\frac{1}{x}$, whence $q=rx$ and the formula becomes $P(1+\frac{1}{x})^{xnr}$, but $(1+\frac{1}{x})^x=e$ at the limit and we have the amount $=Pe^{nr}$ and the interest will be $Pe^{nr}-P$. In this case $P=500$, $e=2.718281828$, $n=10$, $r=.10$, and we have the interest $=500(2.718281828-1)=\$859.1409142$.

II. Solution by W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, Louisiana; and JOHN M. ARNOLD, Crompton, Rhode Island.

$\frac{A}{P}=(1+\frac{r}{q})^{qn}=\frac{A}{500}=(1+\frac{1}{10q})^{10q}$. Expanding the second member, and reducing, $\frac{A}{500}=1+1+\frac{(1-\frac{1}{10q})}{2!}+\frac{(1-\frac{1}{10q})(1-\frac{2}{10q})}{3!}+\dots\dots\dots\dots$

When the intervals are infinitely small the number of intervals (q) is infinitely large, and the fraction in each factor of the numerator of each term is zero. $\therefore \frac{A}{500}=1+1+\frac{1}{2!}+\frac{1}{3!}+\dots\dots\dots\dots$

The sum of this series is the Naperian base. $\therefore \frac{A}{500}=2.718281828$. $\therefore A=1359.140914$, and $A-P=\$859.140914=$ interest required.

III. Solution by Professor J. SCHEFFER, Hagerstown, Maryland.

Let y be the amount, a the initial principal, r the rate per cent., and t the time in years; then, we have from $dy=\frac{rydt}{100}$, $y=Ce^{\frac{rt}{100}}$, but since for $t=0$, $y=a$, we have $C=a$. $\therefore y=ae^{\frac{rt}{100}}$, and interest $=a(e^{\frac{rt}{100}}-1)$. For $r=10$, $t=10$, we have interest $=500(e-1)=500\times1.718281828=\$859.140914$.

Also solved by *O. W. ANTHONY, P. S. BERG, F. P. MATZ, C. D. SCHMITT, H. C. WILKES*, and *G. B. M. ZERR*.

49. Proposed by P. S. BERG, Larimore, North Dakota.

A man having lent \$6000 at 6 per cent. interest *payable quarterly*, wishes to receive his interest in *equal* proportions monthly, and in *advance*. How much ought he to receive monthly?

Let $\$x$=the sum he should receive monthly. But $6000\times.015=\$90=$ quarterly interest. $\therefore$ $1.015x+1.01x+1.005x=\$90$. $\therefore$ $3.03x=\$90$. $x=\$29.70297+$.

Also solved by *P. S. BERG*, *F. P. MATZ*, *J. SCHEFFER*, and *G. B. M. ZERR*.

NOTE.—Solutions of Nos. 46 and 47, Algebra, were received too late for selection from Prof. Benj. F. Yanney, A. M., Mount Union College, Alliance, Ohio.

PROBLEMS.

56. Proposed by D. G. DORRANCE, Jr., Camden, Oneida County, New York.

Sum the series 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, etc., to n terms; also what is the n^{th} term ?

57. Proposed by DAVID EUGENE SMITH, Ph. D., Professor of Mathematics, Michigan State Normal School, Ypsilanti, Michigan.

Prove that the product of the n n^{th} roots of 1 is $+1$ or -1 according as n is odd or even. Prove, and generalize, for the n n^{th} roots of m.

58. Proposed by ROBERT JUDSON ALEY, M. A., Associate Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, California.

Telegraph poles are a yards apart; for how many minutes must one count poles in order that the number of poles counted may be equal to the number of miles per hour that the train is running ?

Solutions of these Problems should be received on or before January 1, 1896.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

45. Proposed by GEORGE E. BROCKWAY, Boston, Massachusetts.

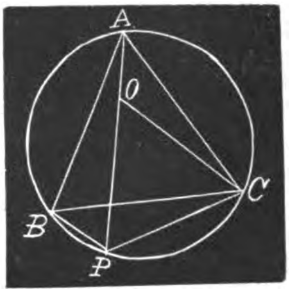
If an equilateral triangle is inscribed in a circle, the sum of the squares of the lines joining any point in the circumference to the three vertices of the triangle is constant.

Solution by JAMES F. LAWRENCE, Breckenridge, Missouri.

Let ABC be the inscribed equilateral triangle and P any point in the circumference of the circle. Join P with the points A, B, and C.

Then $AB^2=BP^2+AP^2-2BP\times AP\cos60^\circ$

$=BP^2+AP^2-BP\times AP$, and



$$AC^2 = CP^2 + AP^2 - 2CP \times AP\cos 60^\circ$$
$$= CP^2 + AP^2 - CP \times AP.$$

$\therefore AB^2 + AC^2 = BP^2 + AP^2 + CP^2 + AP^2 - [BP \times AP + CP \times AP]$.

But $AP \doteq BP + PC$, AMERICAN MATHEMATICAL MONTHLY, Vol. I., No. 9, p. 315, Prob. 19.

$AP^2 = BP \times AP + PC \times AP$, by multiplying both sides of the above equation by AP.

$\therefore AP^2 - [BP \times AP + PC \times AP] = 0$.

$\therefore AB^2 + AC^2 = BP^2 + AP^2 + CP^2$, and $BP^2 + AP^2 + CP^2$ is constant.

Q. E. D.

Excellent solutions of this Problem were received from *P. S. BERG, G. B. M. ZERR, O. W. ANTHONY, COOPER D. SCHMITT, J. F. W. SCHEFFER, JOHN B. FAUGHT, G. I. HOPKINS*, and *E. W. MORRELL*. Two solutions were received without the names of the authors signed to them.

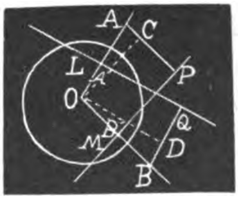
46. Proposed by J. C. GREGG, Superintendent of Schools, Brazil, Indiana.

Given two points A and B and a circle whose center is O: show that the rectangle contained by OA and the perpendicular from B on the polar of A, is equal to the rectangle contained by OB and the perpendicular from A on the polar B.

Solution by JOHN B. FAUGHT, A. B., Instructor in Mathematics, Indiana University, Bloomington, Indiana; P. S. BERG, Larimore, North Dakota; and J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

Let L be the polar of A, and M the polar of B. Let AP be a perpendicular on M, and BA' a perpendicular on L.

Draw OC parallel to M, and OD parallel to L. Then $OA.OA' = OB.OB' = R^2$, by definition.



$$\therefore \frac{OA}{OB} = \frac{OB'}{OA'} = \frac{CP}{BA}.$$

The triangles OAC and OBD are similar.

$$\therefore \frac{OA}{OB} = \frac{AC}{BD} = \frac{CP}{DA} = \frac{AC+CP}{BD+DA} = \frac{AP}{DA}.$$

$\therefore OA.BA = OB.AP$. Q. E. D.

Excellent analytical solutions of this problem were received from *G. B. M. ZERR, COOPER D. SCHMITT*, and *E. W. MORRELL*. Prof. Morrell sent in two solutions.

A solution was also received without the author's name signed to it.

PROBLEMS.

52. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

If the *center* of a rolling ellipse *move in a horizontal line*, determine the surface on which the ellipse rolls.

53. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A pole, a certain length of whose top is painted white, is standing on the side of a hill. A person at A observes that the white part of the pole subtends an angle equal to α and on walking to B, a distance a, directly down the hill towards the foot of the pole the white part subtends the same angle. What is the length of the white part, if the point B is at a distance b from the foot of the pole?

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

22. Proposed by MOSES C. STEVENS, M. A., Professor of Mathematics, Purdue University, Lafayette, Indiana.

Solve the Differential Equation,

$$(6x^3+20x^2-2x)\frac{d^2y}{dx^2}-(9x^2+10x+1)\frac{dy}{dx}+(1+9x)y=0.$$

Solution by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Dividing the given equation by the coefficient of d^2y/dx^2, then representing the coefficient of dy/dx by Q and that of y by Q', we obtain

$$\frac{d^2y}{dx^2}-Q\frac{dy}{dx}+Q'y=0\ldots\ldots(1).$$

In order to reduce (1) to the *typical* form of the text-books $F\left(\frac{dw}{dx},\ \frac{d^2w}{dx^2}\right)$, assume $w=\frac{y}{x+1}\ldots\ldots(a)$; then $\frac{dy}{dx}=(x+1)\frac{dw}{dx}+w$, and $\frac{d^2y}{dx^2}=(x+1)\frac{d^2w}{dx^2}+2\frac{dw}{dx}$.

By substituting in (1), expanding, and reducing, we have

$$\frac{d^2w}{dx^2}=-\left(\frac{2}{x+1}-Q\right)\frac{dw}{dx}\ldots\ldots(2).$$

By putting $dw/dx=p$, (2) becomes

$$\frac{dp}{p}=-\left(\frac{2}{x+1}-Q\right)dx\ldots\ldots(3),$$

in which the variables are *separated*. Integrating each member of (3),

$$\log p, = \log\frac{dw}{dx} = \log\left[\frac{C(3x^2+10x-1)}{x^{\frac{1}{2}}(x+1)^2}\right]\ldots\ldots(4).$$

After passing to exponentials, etc., (4) becomes

$$w = C\int\frac{(3x^2+10x-1)dx}{x^{\frac{1}{2}}(x+1)^2}\ldots\ldots(5).$$

In order to integrate (5), put $x=\tan^2\phi$; then

$$w=2C\left[3\int\sec^2\phi d\phi+4\int d\phi-8\int\frac{d\phi}{\sec^2\phi}\right]=2C\left[3\tan\phi-\frac{4\tan\phi}{\tan^2\phi+1}\right]$$

$$=2C\tan\phi\left[\frac{3\tan^2\phi-1}{\tan^2\phi+1}\right],\quad =3Cx^{\frac{1}{2}}\left[\frac{3x-1}{x+1}\right]+c\ldots\ldots(6).$$

Equating the values of w as given in (a) and (6), we have

$$y=2Cx^{\frac{1}{2}}(3x-1)+c(x+1)\ldots\ldots(7),$$

which is the complete primitive, or *general integral*; and this assertion the following *proof* substantiates.

From (7), $$\frac{y-2Cx^{\frac{1}{2}}(3x-1)}{x+1}=c\ldots\ldots(8).$$

Differentiating (7),

$$\frac{dy}{dx}-C\left(\frac{9x-1}{x^{\frac{1}{2}}}\right)=c\ldots\ldots(9).$$

Differentiating (9),

$$2x^{\frac{3}{2}}\frac{d^2y}{dx^2}\Big/(9x+1)=C\ldots\ldots(10).$$

Eliminating C from (9), by means of (10),

$$\frac{dy}{dx}-2x\left(\frac{9x-1}{9x+1}\right)\frac{d^2y}{dx^2}=c\ \ldots\ldots\ (11).$$

Eliminating C from (8), by means of (10),

$$\frac{y}{x+1}-4x^2\left(\frac{3x-1}{(9x+1)(x+1)}\right)\frac{d^2y}{dx^2}=c\ldots\ldots(12).$$

Equating the values of c as given in (11) and (12),

$$\left(\frac{4x^2(3x-1)-2x(9x-1)(x+1)}{(9x+1)(x+1)}\right)\frac{d^2y}{dx^2}+\frac{dy}{dx}-\frac{y}{x+1}=0\ldots\ldots(13).$$

$$\therefore\ 2x(3x^2+10x-1)\frac{d^2y}{dx^2}-(9x+1)(x+1)\frac{dy}{dx}+(9x+1)y=0,$$

which is the Differential Equation given by the Praposer of the problem.

Scholium.—The proposed Differential Equation is satisfied by the equations, $y=1+x\ldots\ldots(\alpha)$ and $y=2x^{\frac{1}{2}}(3x-1\ldots\ldots(\beta)$; that is, these equations are particular solutions, or *particular integrals*, of the Differential Equation.

36. Proposed by H. C. WHITAKER, B. Sc., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

A cube is revolved on its diagonal as an axis. Define the figure described and calculate its volume.

II. Solution by the PROPOSER.

The edges adjacent to the axis of revolution generate cones; the two edges not adjacent to the axis generate a hyperboloid of one nappe. Take the axis of revolution as the axis of z. The equation of the cones is $x^2+y^2=2(z\pm\frac{1}{3}\sqrt{3}a)^2$, the altitude of each being $\frac{1}{3}\sqrt{3}a$, the radius of each being $\frac{1}{3}\sqrt{6}a$, the volume of each being $\frac{2\pi\sqrt{3}}{27}a^3$.

The equation of the hyperboloid is $2x^2+2y^2-4z^2=a^2$, the volume being the integral of $\pi(2z^2+\frac{a^2}{2})dz$ between the limits $\frac{1}{6}a\sqrt{3}$ and $-\frac{1}{6}a\sqrt{3}$, this volume being $\frac{5\pi\sqrt{3}}{27}a^3$. Adding the volume of the hyperboloid to the volume of the two cones, the total volume is found to be $\frac{\pi}{\sqrt{3}}a^3=1.8138a^3$.

[This solution is given for comparison with that of *DR. ZERR*, published in last issue. EDITOR.]

38. Proposed by L. B. FILLMAN, St. Petersburg, Pennsylvania.

The diameter of the circular base of a dome is $10=a$ feet, which is also the distance from any point on the circumference of the base to any point on the opposite side of the dome from base to apex. Find the volume of the dome.

I. Solution by GEORGE B. McCLELLAN ZERR, M. A., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas; and the PROPOSER.

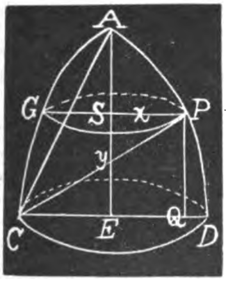
Let $x^2+y^2=a^2$ be the equation to the circle that forms a section of the dome perpendicular to the base. Then any section parallel to the base at distance y from the base has a radius$=(\sqrt{a^2-y^2}-\frac{1}{2}a)$.

$$\therefore\ v=\pi\int_0^{\frac{1}{2}a\sqrt{3}}\left(\sqrt{a^2-y^2}-\tfrac{1}{2}a\right)^2dy=\frac{\pi a^3}{24}(9\sqrt{3}-4\pi),$$

$$=\frac{125\pi}{3}(9\sqrt{3}-4\pi),\ \text{when } a=10,$$

$$=395.59 \text{ cubic feet.}$$

II. Solution by Professor J. SCHEFFER, A. M., Hagerstown, Maryland; and P. S. BERG, Larimore, North Dakota.



$CD=CP=CA=2a$; $AE=\sqrt{4a^2-a^2}=a\sqrt{3}$. Let circular section PG be at distance $ES=y$, $PS=x$; then $(a+x)^2+y^2=4a^2$. $\therefore\ x=-a+\sqrt{4a^2-y^2}$.

$$\text{Capacity}=\pi\int_0^{a\sqrt{3}}x^2dy=\pi\int_0^{a\sqrt{3}}(5a^2-y^2-2a\sqrt{4a^2-y^2})dy$$

$$=\frac{\pi a^3}{3}(9\sqrt{3}-4\pi),\ \text{and for } a=5, \frac{125\pi}{3}(9\sqrt{3}-4\pi).$$

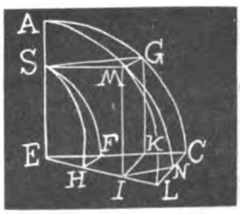
III. Solution by FRANKLIN PIERCE MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Let $AC=PC=DC=r,=10$ feet; $\angle ACD=\phi$, $=60^\circ=\frac{1}{3}\pi$; $\angle PCD=\theta$; $EQ=SP=x$; and $PQ=SE=y$. By well-known principles, then, we have $CE=r\cos\phi$; and, therefore, from the right-angled triangle CPQ, since $CQ=(r\cos\phi+x)$, we obtain the equation $(r\cos\phi+x)^2+y^2=r^2$ ... (1). $\therefore\ x^2=r^2(\cos\theta-\cos\phi)^2$ ......(2). Now, $y=r\sin\theta$. $\therefore\ dy=r\cos\theta d\theta$......(3); also, from Calculus, $V=\pi\int x^2dy$......(4). Transforming (4) by means of (2) and (3), we obtain

$$V=\pi r^3\int_0^{\phi}(\cos\theta-\cos\phi)^2\cos\theta d\theta=\tfrac{1}{12}\pi r^3(\sin3\phi+9\sin\phi-12\phi\cos\phi)\ ......(5),$$

which is undoubtedly the *simplest* general result obtainable. Reducing (5) under the supposition that $\phi=60^\circ=\frac{1}{3}\pi$, we have the well-known result $V=\frac{1}{24}\pi(9\sqrt{3}-4\pi)r^3,=395.59+$ cubic feet.

IV. Solution by W. L. HARVEY, 75 Gray Street, Portland, Maine.



The arc of the dome$=60^\circ$ with radius$=10=a$; hence, the sine$=5\sqrt{3}=\frac{1}{2}a\sqrt{3}$, and this arc revolved about this sine generates the dome.

The quadrant CEA, revolving about EA, generates a hemi-sphere; ESF is a semi-segment of radius $=EA$. If the quadrant revolve a small distance, C

moving to L and G falling on M, and generating $CEALE$, and through MI a plane be passed parallel to CEA, the semi-segment $CGK=EFS$ generates $CKILMG$, of which the part $INLM=EFHS$.

The volume generated by the semi-segment EFS in an entire revolution will equal that generated by CGK *minus* the sum of the solids $CKNIGM$ lying about the circumference of the base of the hemi-sphere. But $GMCKIN=KI\times$ area of the semi-segment CGK; and the sum of all these parts is equal to the circumference described by EK as radius into the same area. If $EA=a$, $EI=c$, $ES=s$, and the arc $SF=p$, we obtain for the solid generated by $EAGK$, $\frac{2\pi}{3}(sc^2+a^3-a^2s)$. Consequently the solid generated by $CGK=\frac{2\pi}{3}(a^2s-sc^2)$.

Then the sum of all the solids $CNIKMG$=semi-segment $GCK\times 2\pi c=\pi(cap-sc^2)$, and the volume sought is $\frac{2\pi}{3}(a^2s-sc^2)-\pi(cap-sc^2)=\pi(\frac{sc^2}{3}+\frac{2a^2s}{3}-cap)$. Putting $c^2=a^2-s^2$ this becomes, $\pi(sa^2-\frac{s^3}{3}-cap)$. In the problem $s=\frac{a}{2}\sqrt{3}$, $c=\frac{a}{2}$, $p=a(60^\circ)=\frac{a\pi}{3}$. Then $cap=\frac{a^3\pi}{2.3}$, and the contents are $\frac{a^3\pi}{2}\left(\frac{3\sqrt{3}}{4}-\frac{\pi}{3}\right)=395.59$ cubic feet. This method of solution was suggested by a solution of a similar problem by Professor Seyford, of Colby University.

[From the MONTHLY of October, 1894, pp. 257-8, we have *four* other different solutions of a similar problem; and *each* solution gives the result, $V=\frac{1}{3}\pi(9\sqrt{3}-4\pi)a^3$, $\frac{125}{3}\pi(9\sqrt{3}-4\pi)=395.59027+$ cubic feet. EDITOR.]

39. Proposed by J. C. GREGG, Brazil, Indiana.

Show that the curve

$$x=9a\sin\theta-4a\sin^3\theta$$
$$y=-3a\cos\theta+4a\cos^3\theta$$

is symmetrical to the axes, and has double points and cusps: find the lengths of the arcs, (a) between the double points, (b) between a double point and a cusp, (c) and the arc connecting two cusps and not passing through a double point. [*Johnson's Calculus.*]

Solution by GEORGE B. McCLELLAN ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

The equations as given in *Johnson's Calculus* are

$x=9a\sin\theta-4a\sin^3\theta=6a\sin\theta+a\sin3\theta$,

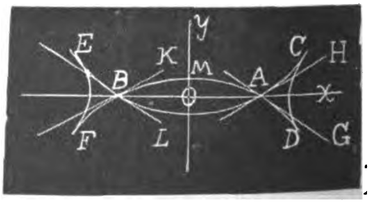
$y=-3a\cos\theta+4a\cos^3\theta=a\cos3\theta$.

$\therefore r^2=x^2+y^2=a^2+72a^2\sin^2\theta-48a^2\sin4\theta$;

$dx=6a\cos\theta+3a\cos3\theta$, $dy=-3a\sin3\theta$.

$ds=\sqrt{dx^2+dy^2}=3a(4\cos^2\theta-1)$.

$\therefore s=3a(\theta+\sin2\theta)$.



$$\frac{dy}{dx} = -\frac{\sin 3\theta}{2\cos\theta + \cos 3\theta} = \frac{0}{0}, \text{ when } \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}.$$

$\therefore$ There are cusps at C, D, E, F. When $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$, $x = 4a$, $y = 0$, $dy/dx = \mp\frac{1}{\sqrt{3}}$; when $\theta = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$, $x = -4a$, $y = 0$, $dy/dx = \pm\frac{1}{\sqrt{3}}$.

$\therefore$ A and B are double points. The curve with the tangents at its double points is given in the figure. It is symmetrical to the axes, has four cusps and two double points.

(*a*), $s = AMB = 3a\left[\theta + \sin 2\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = 3a\left(\frac{\pi}{3} + \sqrt{3}\right) = a(\pi - 3\sqrt{3})$,

(*b*), $s = AD = 3a\left[\theta + \sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{2}\pi a$,

(*c*), $s = DC = 3a\left[\theta + \sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}} = a(3\sqrt{3} - \pi)$.

Whole length of curve $= 2AMB + 4AD + 2CD = 2a(\pi + 6\sqrt{3})$.

[The given curve is one of the involutes of a 4-cusped hypocycloid, which could be drawn surrounding the figure. The curve as given by the equations as first proposed is symmetrical to the axes, has two double points but no cusps. EDITOR.]

PROBLEMS.

47. Proposed by Professor J. SCHEFFER, A. M., Hagerstown, Maryland.

The floor of a vault forms a square, and all sections parallel to it are squares. The two vertical sections through the middle points of the opposite sides of the floor are equal semi-circles. Find the convex surface and the volume of the vault.

48. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

I have a circular section basin 12 inches in perpendicular height; the diameters are as follows: At base, 2 inches; one inch perpendicular height, 6 inches; two inches perpendicular height, 18 inches; three inches perpendicular height, 54 inches; and so on, the diameter being trebled for every inch in height. After a rain the water in the basin is six inches deep, what was the rainfall?

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

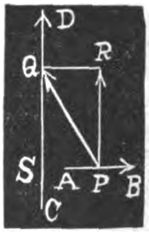
SOLUTIONS OF PROBLEMS.

28. Proposed by O. W. ANTHONY, Professor of Mathematics, New Windsor College, New Windsor, Maryland.

A movable finite wire carrying a current of electricity is perpendicular to and on one side of an infinite wire also carrying a current. Investigate the motion of the movable wire.

Solution by the PROPOSER.

Let AB be the finite wire, and DC the infinite wire. Let the current flow away from the infinite wire in the short one. Also call μ_1, μ_2 the current strengths of the two currents, and m the power of the finite wire. Then $df_{PQ}=\mu_1\mu_2\frac{dxdz}{x^2+z^2}$, $x=QS$, $z=PS$. Resolving forces perpendicular and parallel to DC we have



$$df_{PR}=\mu_1\mu_2\frac{xdxdz}{(x^2+z^2)^{\frac{3}{2}}}.$$

$$\therefore f_{PR}=2\mu_1\mu_2\int_{z_1}^{z_2}\int_0^{\infty}\frac{xdxdz}{(x^2+z^2)^{\frac{3}{2}}}.$$

$$=2\mu_1\mu_2\log\left(\frac{z_2}{z_1}\right).$$

$$\therefore \frac{d^2s}{dt^2}=\frac{2}{m}\mu_1\mu_2\log\left(\frac{z_2}{z_1}\right).$$

$$s=\frac{1}{m}\mu_1\mu_2\log\left(\frac{z_2}{z_1}\right)t^2+k_1t+k_2.$$

29. Proposed by J. A. CALDERHEAD, A. B., Superintendent of Schools, Limaville, Ohio.

Show that if a body be projected from the angle A of a plane triangle ABC so as to strike the side CB at a point D, then, if its course after reflection at D be parallel to AB, $\tan\angle DAB=\frac{(1+e)\cot B}{1-e\cot^2 B}$.

Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics in the University of Mississippi, University P. O., Mississippi.

The angle between the course of the body before impact and the side CB is $180^\circ-(B+DAB)$.

$$\therefore e=\frac{\tan B}{\tan[180^\circ-(B+DAB)]};$$

$$-\tan B = e\,\frac{\tan B + \tan A}{1 - \tan B\,\tan A};$$

$$\tan DAB = \frac{(1+e)\tan B}{\tan^2 B - e}$$

$$= \frac{(1+e)\cot B}{1 - e\cot^2 B}.$$

Also solved by *O. W. ANTHONY*, and *J. SCHEFFER.*

PROBLEMS.

35. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

A man weighs 150 pounds; his balloon with all its attachments weighs 500 pounds. What volume of pure hydrogen must be made and put into the balloon so that it will be on the point of ascending with the man? How many kilograms of zinc and of hydrogen sulphate will be used in generating the hydrogen? Give volume of hydrogen in cubic feet given that one litre of hydrogen weighs .0896 grams.

36. Proposed by O. W. ANTHONY, Professor of Mathematics, New Windsor College, New Windsor, Maryland.

A vertical slit is made in the middle of the side of a rectangular box containing water. What is the time required to empty the box?

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

26. Proposed by J. W. WATSON, Middlecreek, Ohio.

Find the average area of all right-angled triangles having a *constant* hypotenuse.

III. Solution by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

In my first two solutions I made an arm of the right-angled triangle vary uniformly, although I employed two different systems of co-ordinates. I continued this variation until the arms of the right-angled triangle became equal; and by doing this I avoided all *reciprocal* equal right-angled triangles. Looking at results from this standpoint, the verdict must be—*correct*. By simply varying uniformly an arm of an inscribed right-angled triangle, as I have done, I am now fully convinced that not all the *possible* right-angled triangles are comprehended; for, most certainly, the uniform variation of an arm does not cause a uniform variation of the vertex of the right angle—along a quadrant of the circumscribing circle. I say the *most natural* solution of this problem is the one in which the *arc* of the circumscribing circle is made to vary uniformly, and this variation is to extend over only one quadrant of the circumscribing circle. Let x=the arc intercepted by an arm and the constant hypotenuse h; then, if $\frac{1}{4}\pi h=a$, the required average area is

$$A=\tfrac{1}{2}h^2\int_0^a \sin\left(\frac{x}{h}\right)\cos\left(\frac{x}{h}\right)dx\div\int_0^a dx=\frac{h^2}{2\pi}\ldots\ldots(1).$$

FOURTH SOLUTION.

Taking θ as the *central* angle intercepted by an arm, and by the constant hypotenuse, of the right-angled triangle, the required average area becomes

$$A=\tfrac{1}{4}h^2\int_0^{\frac{1}{2}\pi}\sin\theta d\theta\div\int_0^{\frac{1}{2}\pi}d\theta=\frac{h^2}{2\pi}\ldots\ldots(2).$$

FIFTH SOLUTION.

Making ϕ one of the acute angles, the required average area becomes

$$A=\tfrac{1}{4}h^2\int_0^{\frac{1}{2}\pi}\sin2\phi d\phi\div\int_0^{\frac{1}{2}\pi}d\phi=\frac{h^2}{2\pi}\ldots\ldots(3).$$

SIXTH SOLUTION.

Let the origin of Cartesian co-ordinates be placed at the center of the circle; then the required average area becomes

$$A=\tfrac{1}{4}h^2\int_0^{\frac{1}{2}h}dx\div\tfrac{1}{2}h\int_0^{\frac{1}{2}\pi}\frac{dx}{\sqrt{(\frac{1}{4}h^2-x^2)}}=\frac{h^2}{2\pi}\ldots\ldots(4).$$

Several other solutions leading to the same result are possible.

NOTE.—Professor O. W. Anthony sent us a note in which he defends the solution leading to $\frac{1}{6}a^2$ as the answer. His argument being, in substance, this: The mind does not form a picture of a right triangle inscribed in a semi-circle whose diameter is a but simply a right triangle whose hypotenuse is a. He therefore concludes that the number of triangles should be found by varying one of the sides.

The discussion of this problem called forth the excellent article, "A Note on Mean Values," by Dr. Moore, page 303 of this issue of the MONTHLY. I am quite sure that that article will be greatly appreciated by those of our readers who are interested in this abstruce subject, Mean Value.

Taking the substance of that article as criterion, it remains to determine whether or not the above problem is stated in the *definite* form. I hold the opinion that it is stated in the *definite* form; for the problem requires the average area of *all* right triangles having a given hypotenuse. It does not require the average area of all right triangles having a given hypotenuse and formed according to a certain law, but all the right triangles having a given hypotenuse and the law of formation must be so chosen as to give *all* such right triangles. Therefore, the solutions leading to the result $\frac{a^2}{2\pi}$ are the *correct* and *only* solutions of the problem. EDITOR.]

PROBLEMS.

35. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

Find the chance that the distance of two points within a square shall not exceed a side of the square. [*Byerly's Integral Calculus, p. 209.*]

36. Proposed by O. W. ANTHONY, Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

A box contains n^2 blocks numbered from 1 to n^2. What is the probability that two consecutive numbers will be adjacent ?

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

DIOPHANTUS' EPITAPH.

Translated by Rev. WRIGHT G. CAMPBELL, A. M., Professor of Ancient Languages, Irving College, Mechanicsburg, Pennsylvania.

Here Diophantus has a mound which to you, with wonderful art, signifies the times of his life.

One-sixth part he lived a youth ; then, in the twelfth part, slowly he began to clothe his cheeks.

In the seventh part after these, he was joined to a wife ; and in the fifth year, a beautiful boy was born.

After he had attained half of the paternal age, he unhappy seized by sudden death, died.

Four summers the surviving father was compelled to mourn.

From this you may ascertain his years. *Contributed by F. P. Matz.*

SOLUTIONS OF PROBLEMS.

NOTE.—The comment on page 285 of last issue should have been credited to M. A. Gruber, War Department, Washington, D. C.

27. Proposed by J. W. NICHOLSON, LL. D., President, and Professor of Mathematics, Louisiana State University and A. and M. College, Baton Rouge, Louisiana.

Required a formula for finding five integers the sum of whose cubes is a cube.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

A very simple formula is obtained as follows:

$x^3+(6x)^3+(8x)^3=(9x)^3$......(1); $(3x)^3+(4x)^3+(5x)^3=(6x)^3$......(2).

(2) in (1) gives $x^3+(3x)^3+(4x)^3+(5x)^3+(8x)^3=(9x)^3$, where x can have any value, positive, integral.

II. Solution by H. W. DRAUGHON, Olio, Mississippi.

If, in the identity, $(x+y)^3=x^3+3x^2y+3xy^2+y^3$......(1), we make $3x^2y$ a cube, and $3xy^2$, the sum of two cubes, we will have a formula for finding five numbers the sum of whose cubes is a cube.

Put $3x^2y=m^3x^3$; whence $x=\frac{3y}{m^3}$, and $3xy^2=\frac{9y^3}{m^3}=\left(\frac{2y}{m}\right)^3+\frac{y^3}{m^3}$.

By substitution, (1), becomes,

$$\left(\frac{3y}{m^3}+y\right)^3=\left(\frac{3y}{m^3}\right)^3+\left(\frac{3y}{m^2}\right)^3+\left(\frac{2y}{m}\right)^3+\left(\frac{y}{m}\right)^3+y^3.$$

Multiplying out by m^9, we have

$$(3+m^3)^3y^3=27y^3+27m^3y^3+8m^6y^3+m^6y^3+m^9y^3 \ldots\ldots (A),$$

which is the formula required. Let $y=1$, and $m=4$; then we have,

$$(3)^3+(12)^3+(32)^3+(16)^3+(64)^3=(67)^3.$$

III. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

Let x_1, x_2, x_3, x_4, x_5, be the integers. Then

$$x_1^3+x_2^3+x_3^3+x_4^3+x_5^3=x^3 \ldots\ldots (1).$$

Let $x_1=3m$, $x_2=4m$, $x_3=5m$. Then (1) may be written

$$(6m)^3+x_4^3+x_5^3=x^3 \ldots\ldots (2), \text{ or } z^3+x_4^3+x_5^3=x^3 \ldots\ldots (3).$$

Let $z=3u$, $4u$, or $5u$, $x_4=4u$, $5u$, or $3u$, $x_5=5u$, $3u$, or $4u$.

We have $6m=3u$; $6m=4u$; $6m=5u$. Then $u=2m$; $u=\frac{3m}{2}$; $u=\frac{6m}{5}$.

Therefore,

$$\left.\begin{aligned} x_1&=3m,\\ x_2&=4m,\\ x_3&=5m,\\ x_4&=8m,\\ x_5&=10m,\end{aligned}\right| \text{I.} \qquad \text{Or} \qquad \left.\begin{aligned} x_1&=3m,\\ x_2&=4m,\\ x_3&=5m,\\ x_4&=\frac{15m}{2},\\ x_5&=\frac{9m}{2},\end{aligned}\right| \text{II.}$$

$$\text{Or} \qquad \left.\begin{aligned} x_1&=3m,\\ x_2&=4m,\\ x_3&=5m,\\ x_4&=\frac{18m}{5},\\ x_5&=\frac{24m}{5},\end{aligned}\right| \text{III.}$$

Changing II, III, to a form which shall always be integral, we have the following table of formulæ :

$$\begin{aligned} x_1&=3m,\ 6m,\ 15m.\\ x_2&=4m,\ 8m,\ 20m.\\ x_3&=5m,\ 10m,\ 25m.\\ x_4&=8m,\ 15m,\ 18m.\\ x_5&=10m,\ 9m,\ 24m.\end{aligned}$$ Where m has any integral value.

28. Proposed by DAVID EUGENE SMITH, Ph. D., Professor of Mathematics, Michigan State Normal School, Ypsilanti, Michigan.

Decompose the product 97.673.257 into the sum of two fourth powers.

I. Solution by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

Solution by Determinants :

$$97=\begin{vmatrix} 9 & -4\\ 4 & 9\end{vmatrix},\quad 673=\begin{vmatrix} 23 & -12\\ 12 & 23\end{vmatrix},\quad 257=\begin{vmatrix} 16 & -1\\ 1 & 16\end{vmatrix}.$$

By multiplication $673.97=\begin{vmatrix} 23 & -12\\ 12 & 23\end{vmatrix}\times\begin{vmatrix} 9 & -4\\ 4 & 9\end{vmatrix}=\begin{vmatrix} 255 & -16\\ 16 & 255\end{vmatrix}.$

and $\begin{vmatrix} 255 & -16\\ 16 & 255\end{vmatrix}\times\begin{vmatrix} 16 & -1\\ 1 & 16\end{vmatrix}=\begin{vmatrix} 4080-16, & 255-256\\ 256-255, & 16+4080\end{vmatrix}$

$$=\begin{vmatrix} 4096 & -1\\ 1 & 4096\end{vmatrix}=\begin{vmatrix} 64^2 & -1\\ 1 & 64^2\end{vmatrix}=64^4+1^4.$$

II. **Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.**

$$97.673.257=(3^4+2^4)(23^2+12^2)(4^4+1^4).$$

$$(a^4+b^4)(c^4+d^4)=(a^2c^2\pm b^2d^2)^2+(a^2d^2\mp b^2c^2)^2=A^2+B^2.$$

$$(a^4+b^4)(c^4+d^4)(e^2+f^2)=(A^2+B^2)(e^2+f^2)=(Ae\pm Bf)^2+(Af\mp Be)^2$$

$$=\{(a^2c^2\pm b^2d^2)e\pm(a^2d^2\mp b^2c^2)f\}^2+\{(a^2c^2\pm b^2d^2)f\mp(a^2d^2\mp b^2c^2)e\}^2.$$

Let $a=3$, $b=2$, $c=4$, $d=1$, $e=23$, $f=12$.

$\therefore$ $97.673.257=1^2+4096^2=3359^2+2344^2=3041^2+2744^2=1511^2+4064^2$.

Of these four sums, the first is the only one that fulfills the condition.

$\therefore$ $97.673.257=1^4+64^4$

NOTE.—*A. H. BELL, H. W. DRAUGHON, J. C. CORBIN,* and *F. P. MATZ* should have been credited with solutions of No. 22.

PROBLEMS.

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40. **Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.**

The sum of the three positive integral *cubic* roots of an equation is a square. What is the equation ?

41. **Proposed by H. C. WILKES, Skull Run, West Virginia.**

Given $\dfrac{50(a+b)}{ab}=\dfrac{81(c+d)}{cd}$......(1) ; $\dfrac{56(a+c)}{ac}=\dfrac{75(b+d)}{bd}$......(2); $\dfrac{65(b+c)}{bc}=\dfrac{66(a+d)}{ad}$......(3), to find the least integral values of a, b, c, d.

42. **Proposed by E. B. ESCOTT, 6123 Ellis Avenue, Chicago, Illinois.**

In a parallelogram, sides a and b, diagonals c and d, $2a^2+2b^2=c^2+d^2$. Find all the parallelograms, not rectangles, whose sides and diagonals are rational.

Examples :

a	*b*	*c*	*d*
4	7	9	7
16	7	21	13
8	9	13	11
8	11	17	9

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

28. Proposed by "IAGO"—(The late DR. JAMES MATTESON, DeKalb Center, Illinois.)

If 9 gentlemen, or 15 ladies, will eat 17 apples in 5 hours, and 15 gentlemen and 15 ladies can eat 47 apples of a similar size in 12 hours, the apples growing uniformly; how many boys will eat up 360 apples in 60 hours, admitting that 120 boys can eat the same number as 18 gentlemen and 26 ladies? *F. P. Matz.*

Solution by Professor P. S. BERG, Larimore, North Dakota.

1. Call the original size of an apple an "apple unit."
2. Call the growth of 1 apple in 1 hour one "unit of growth."
3. $32\frac{1}{7}$ boys will in the same time eat as many as 9 gentlemen or 15 ladies.
4. $\therefore$ $32\frac{1}{7}$ boys in 5 hours or $160\frac{5}{7}$ boys in 1 hour$=17$ "apple units"$+85$ "units of growth."
5. $\therefore$ $85\frac{5}{7}$ boys in 12 hours or $1028\frac{4}{7}$ boys in 1 hour$=47$ "apple units"$+$ 564 "units of growth."
6. $\therefore$ $(4)\times2\frac{13}{17}=444\frac{39}{119}$ boys in 1 hour$=47$ "apple units"$+235$ "units of growth."
7. $\therefore$ $(4)\times6\frac{54}{85}=1066\frac{46}{119}$ boys in 1 hour$=112\frac{4}{5}$ "apple units"$+564$ "units of growth."
8. $\therefore$ $(5)-(6)=584\frac{29}{119}$ boys in 1 hour$=329$ "units of growth."
9. $\therefore$ $(7)-(5)=37\frac{97}{119}$ boys in 1 hour$=65\frac{4}{5}$ "apple units."
10. $\therefore$ $(9)\times15\frac{9}{20}=584\frac{29}{119}$ boys in 1 hour$=1016\frac{61}{100}$ "apple units."
11. $\therefore$ 329 "units growth"$=1016\frac{61}{100}$ "apple units,
    1 "unit growth"$=3\frac{9}{100}$ "apple units."
12. 360 apples in 60 hours$=360$ "apple units"$+21600$ "units growth."
13. 360 "apple units"$=116\frac{156}{309}$ "units growth."
14. $\therefore$ 360 apples in 60 hours$=116\frac{156}{309}+21600=21716\frac{156}{309}$ "units growth."
15. From (8) 1 boy in 1 hour$=\frac{119\times329}{69525}$ "units growth."
16. $\therefore$ 1 boy in 60 hours$=\frac{156604}{4635}$ "units growth."
17. $\therefore$ $21716\frac{156}{309}\div\frac{156604}{4635}=643$ boys.

29. Proposed by ALEXANDER MACFARLANE, M. A., D. Sc., LL. D., Professor of Electrical Engineering in Lehigh University, South Bethlehem, Pennsylvania.

A rectangular room has the four walls, the floor, and the ceiling covered with mirrors; a candle is placed inside the room: find a formula which will express all the images.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Regard the candle as a luminous point. Then, since there are three sets of parallel mirrors, we have, from elementary optics, three sets of an infinite number of images situated on three straight lines at right angles to one another, and intersecting at the bright point.

These mirrors are also inclined to one another at right angles. Let $A^\circ=\frac{\pi}{2}$=the angle of inclination of the mirrors, a°, b° the angles made by the candle with two of the mirrors.

Then $\frac{360^\circ-(a^\circ+b^\circ)}{A^\circ}=\frac{360^\circ-90^\circ}{90^\circ}=3$=the number of images due to two of the mirrors inclined at 90°. There are twelve such sets of inclined mirrors, but of the 36 images formed, 18 are repeated. $\therefore$ ½ of 12 of 3=18 images due to the inclined mirrors.

$\therefore \frac{12\{2\pi-(a^\circ+b^\circ)\}}{\pi}$, is the formula for the images due to the inclined mirrors, where $a^\circ+b^\circ=\frac{\pi}{2}$.

30. **Proposed by R. J. ADCOCK, Larchland, Warren County, Illinois.**

When the sum of the distances of a point of a plane surface, from all the other points, is a minimum, that point is the center of gravity of the plane surface.

I. **Proof by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.**

Let (x_1, y_1) be the point, (x, y) any other point, S the sum of the distances of (x, y) from (x_1, y_1).

$$\text{Then } S=\int_{x_1}^{x_2}\int_{y_1}^{y_2}\sqrt{(x-x_1)^2+(y-y_1)^2}\,dxdy.$$

Let $\int_{x_1}^{x_2}\int_{y_1}^{y_2}$ be represented by $\int$, and $dxdy$ by dA.

$$\therefore S=\int\sqrt{(x-x_1)^2+(y-y_1)^2}\,dA=\int D\,dA.$$

$$\text{For a minimum, } \frac{dS}{dx_1}=\frac{(x-x_1)dA}{D}=0,\ \frac{dS}{dy_1}=\frac{(y-y_1)dA}{D}=0.$$

$$\therefore (x-x_1)dA=0,\ (y-y_1)dA=0.\quad \because x_1=\frac{\int xdA}{\int dA},\ y_1=\frac{\int ydA}{\int dA}.$$

$$\therefore x_1=\frac{\int\int xdxdy}{\int\int dxdy},\ y_1=\frac{\int\int ydxdy}{\int\int dxdy}.$$

II. **Remark by S. H. WRIGHT, M. D., M. A., Ph. D., Penn Yan, New York.**

Mr. Adcock's problem asserts the truth evidently, when regular plane surfaces are considered, such as the square, rectangle, parallelogram, rhombus, the circle, and *equilateral polygons*. I hardly believe the problem will apply to *any irregular* figure.

III. **Comment, etc., by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.**

It is evidently meant that when the sum of the squares of the distances of a point from all other points is a minimum the point is the *c. g.* of the surface. It can easily be proved that the other is not true.

[If Prof. Anthony will furnish a proof that the proposition does not hold for *any or all figures* we will be glad to publish it. We append Prof. Anthony's proof of the well-known proposition which he quotes. EDITOR.]

[The sum of the squares of the distances of a point (h, k) from all other points in the surface is $u=\int\int\Big[(x-h)^2+(y-k)^2\Big]dxdy$, where the integration is over the entire surface. For minimum, $\dfrac{du}{dh}=0$, $\dfrac{du}{dk}=0$. i. e.,

$$\int\int(x-h)dxdy=0, \text{ and} \int\int(y-k)dxdy=0;$$

$$\text{Whence } h=\frac{\int\int xdxdy}{\int\int dxdy}, \text{ and } k=\frac{\int\int ydxdy}{\int\int dxdy}.$$

That is (h, k) is the center of gravity of the surface.]

NOTE. In Prof. Ross' problem in September-October No., p. 291, read "*square* field *ABC*" instead of "*rectangular* field;" also insert "irregular" before the second "plane curve" in line 2 of Prof. Taylor's problem, and read "distance" for "distances" and $(C)=h$ for $(C=h)$ in line 5 of same problem.

PROBLEMS.

35. **Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy in Pacific College, Santa Rosa, California; P. O., Sebastopol, California.**

To an observer whose latitude is 40 degrees north, what is the sidereal time when Fomalhout and Antares have the same altitude: taking the Right Ascension and Declination of the former to be 22 hours, 52 minutes, —30 degrees, 12 minutes; of the latter 16 hours, 23 minutes, —26 degrees, 12 minutes?

36. **Proposed by J. K. ELLWOOD, A. M., Principal of the Colfax School, Pittsburg, Pennsylvania.**

"What is the length of a chord cutting off one-fifth of the area of a circle whose diameter is 10 feet?"

37. **Proposed by F. M. SHIELDS, Coopwood, Mississippi.**

A gentleman owned and lived in the center, *R*, of a rectangular tract of land whose diagonal, *D*, 350 rods, dividing the tract into two equal right-angled triangles, in each of which is inscribed the largest square field, *F* and *F*, possible; the north and south boundary lines of the two square fields being extended and joined formed a little rectangular lot, *R*, in the center around the residence. The difference in the area of the *entire rectangular* tract and the *sum* of the areas of the two square fields, *F*, *F*, is 187½ acres. Give the dimensions and area of the entire tract, and one square field, *F*.

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

A POSTULATE OF THE HYPOTHESIS OF THE FOURTH DIMENSION.

Let it be granted that a straight line may be drawn through any point of the space in which our universe is contained, every other point of the supposed straight line being outside of our space.

This is *a postulate* logically involved in Arthur Willink's speculation respecting a fourth direction. He says that the fourth direction is *unknown*. He reasons that this could not be if two points in the fourth direction were posited in our space, since two given points in a straight line determine its position, and its direction becomes known.

According to Arthur Willink the direction of the fourth dimension is *unknown*. Hence, the fourth dimension can intersect our space in but *one* point. Hence, every other point on the hypothetical fourth dimensional straight line except that of the intersection must lie outside of our trinally extended space. Practically to the denizens of our universe that means that a straight line may be drawn where our space is not. This hypothesis of a fourth dimension, therefore, places restrictions upon the extent of our space, whereas no ultimate boundary is assigned to it by either the intellect or the imagination of man. The human mind reports as the result of its cognition *one illimitable space*. The hypothesis of another and a wider space is inconsistent with this cognition.

Let us view this subject in another light. Three straight lines mutually perpendicular to each other may be drawn through any point in our space, and hence through the point in which the fourth dimension is supposed to intersect it. The third dimension is perpendicular to the plane of the first and second dimensions. If this plane is definitely located, the direction of the third dimension is determined. Is the fourth dimension, also, at right angles to this plane? If so it must coincide with the third dimension and therefore lie in our space. But this conclusion contradicts the hypothesis that the fourth dimension is not in our space.

Finally, if the alleged fourth dimension is not perpendicular to the plane of the first and second dimensions is it a *legitimate* dimension?

JOHN N. LYLE.

The derivitive of $e^{-xi}(\cos x+i\sin x)$ is zero.

$\therefore e^{-xi}(\cos x+i\sin x)=A$, a constant.

Putting $x=0$, $A=1$.

$\therefore e^{xi}=\cos x+i\sin x$.

WM. E. HEAL, Marion, Indiana.

In $e^{xi}=\cos x+i\sin x$, let $x=m\pi$. $\therefore e^{m\pi i}=\cos m\pi+i\sin m\pi$. $\therefore m\pi i=\log[\cos m\pi+i\sin m\pi]$. If in this m is any positive or negative integer. the last term, $i\sin m\pi$, will disappear; and we may write: $m\pi i=\log\cos m\pi$. Now for any *even* number form, $\cos m\pi$ becomes $+1$, for any *odd* -1. $\therefore \log\cos m\pi=\log(\pm1)$. $\therefore m\pi i=\log(\pm1)$. For $m=1$, $\pi i=\log(-1)$. We may then write: $m\log(-1)=\log(\mp1)$ [Cf. *Schurig's Algebra*, 73, 2.z.]. The upper sign is to be taken for m even, the lower for m odd.

If $u=e^x$, $\log(\pm u)=\log u+\log(\pm1)=\log e^x+\log(\pm1)=x+m\log(-1)$. The sign to be taken as before.

OSCAR SCHMIEDEL, A. M., Professor of Mathematics,
Bethany College, Bethany, West Virginia.

WANTED.—A solution by Quaternions, giving the elementary steps in simple English, of some problem published in the MONTHLY.

A READER.

Olney, in his *General Geometry*, says "The problem of the duplication of the cube and the trisection of an angle has been shown to be identical, as both depend upon the insertion of two means in a continued proportion between two extremes." Where can I find a proof of the statement in reference to the trisection of an angle?

N. F. DAVIS, 21 George Street,
Providence, Rhode Island.

(A) The trisection of an angle: *The trisection of a right line taken as the chord of any arc of a circle trisects the angle of the arc;* (B) Duplication of the Cube: *Doubling the dimensions of a cube octuples its contents, and doubling its contents increases its dimensions twenty-five plus one per cent.*

By request of the author,
EDW. J. GOODWIN, Solitude, Indiana.

DEFINITION OF A FRACTION.

I. In compliance with your request in September No. (1894) of MONTHLY, I state that my preference of definitions of fractions is (B).

H. W. DRAUGHON.

II. I think that a fraction is an indicated division. Division has two meanings: 1, The division into parts; 2, The division by a number of the same kind. A group of 12 cows÷4=a group of 3 cows. 12 cows÷4 cows=3 times as many.

H. C. WHITAKER.

III. The definition (A) suits me the best if only applied to "abstract" arithmetic—provided, the meaning of "a unit" is defined in advance. But as the sentence reads: "a unit or anything else," it leaves a suspicion in my mind that all mathematicians have not a fixed definition for a unit in the scientific sense. To get at what a mathematical fraction is, we must first all agree on a common basis for numeric systems. I submit my views. First, notation of any kind, or the symbols (figures) we call numbers, have *no value other* than that given by the "something" the numbers represent; second, every numerical system is founded on three principles, viz: singularity, plurality, and totality. The first principle is represented by the so-called *indefinite* one. All indefinite ones represent, in the scientific sense, indivisible fractions, of utility only in counting "items." From these indivisible ones *units* are raised, that is, the least "plural" (the second principle) is obtained by composing two of the indivisible ones into one united whole. This, the first and lowest unit, is called two. Again, by uniting the lowest unit and the indivisible fraction into a greater whole, called three, the least numeric total (the third principle) is expressed.

According to this conception of numbers, a multitude of fractions are convertible into units of divers magnitude and from a number of units, a multitude of fractions or finite ones can be evolved, but every unit used in computation *must have a fixed, definite, intrinsic* value (its magnitude) as well as a relative value depending on its combination with other units and fractions.

With this understanding of what constitutes the foundation for all scientific numerical systems (not applicable to empiric systems), the definition of a fraction may be reduced to this:—A fraction is a given part of a given unit the magnitude of which is called the denominator, and is written under a line, while the given part or parts of the unit's magnitude is written above the line and is called the numerator; hence, the numerator indicates the fraction of a certain unit designated by the denominator.

CHAS. DE MEDICI,
60 West 22d Street, New York City.

[We give all our subscribers opportunity to express *their* views *briefly* upon the questions raised in the MONTHLY whether these views are founded upon what we consider right conceptions or not. EDITORS.]

NOTE ON HELMHOLTZ'S USE OF THE TERMS "SURFACE" AND "SPACE" AS IDENTICAL IN MEANING:—Does the "immortal" Helmholtz in his Lectures on the—"Origin and Significance of Geometrical Axioms"—use the terms "surface" and "space" as identical in meaning?

If so, is not his performance "plainly pseudological" however "sickening" the statement of the fact may be to the devotees who would have us bow down before their pseudo-spherical idol and reverence their pseudological prophet as an infallible authority.

JOHN N. LYLE.

COMMENTS ON DUREGE'S THEORY OF FUNCTIONS.

Durege's book is admirably adapted for introducing beginners to the theory of functions according to the methods of Riemann. Ever since its first appearance, I have been in the habit of recommending it to my students.

PROFESSOR DR. L. FUCHS,
September 5th, 1895. University of Berlin.

His (Durege's) success lies in his art of execution; the material chosen with rare skill is worked up into a continuous whole and treated with masterly lucidity, and the interest is constantly kept up by happy examples. The present work, the fourth edition, which was prepared by Durege shortly before his death, is excellently adapted for an introduction to the study of complex variables. Beginning with the historical development of the idea of complex quantities, the author first establishes the general properties of functions of such quantities. * * * Durege's book will certainly maintain for a long time to come the prominent position in literature which it occupies at present.

Zeitschript fur das Realschulwesen, 1894.

An instructive book, in which is developed according to the ideas of Cauchy, and especially of Riemann, the classical theory of a complex or imaginary variable. The examples are simple and well chosen. * * * A good manual for students.

Revue Generale des Sciences Pures et Appilquces, 1894.

A new edition of Dr. Durege's most excellent book, well known to every mathematician, lies before us. Compared with the third edition, only a few changes appear, consisting in short additions, more numerous examples, and alterations in style. * * * The excellencies of Durege's presentation of the elements of the theory of functions are so generally known, that they relieve the reviewer in a most agreeable way of making any commendation or criticism. We have no book which is so well adapted for the introduction to the more recent theory of functions, and for facilitating the study of the works of that great mathematician (Riemann), as the one here noticed.

Naturwissenschaftliche Wochenschript, 1894.

The fourth edition of the present work has now appeared, which fact is a conclusive proof of its excellence. As this edition, compared with the third, contains only a few changes, consisting in short additions, more numerous examples, alterations in style, etc., we may dispense with a more minute discussion of it. The work is based upon the modern theories of functions following Riemann's method, and is eminently adapted to introduce the student to the conceptions of this great mathematician—whom it is quite difficult to understand in the original. The present work has full raison d' etre of Neumanns book "Vorlesurigen uber Riemann's Theorie der Abelschen Integrale," and it is even to be recommended to a beginner in preference to the latter on account of its greater precision.

Central Organ fur die Interessen des Realschulwesens, 1894.

EDITORIALS.

No. 10, Vol. II, of the *Mathematical Magazine* is expected to be ready in January, 1896.

Prof. William Hoover should have been credited with solving Problem 17, Average and Probability. His solution was accidentally mislaid.

D. G. Durrance sent in a solution of Problem 49, Department of Arithmetic, after the July-August number had gone to press. His solution is by Algebra.

Drs. W. W. Beman and D. E. Smith have written a *Plane and Solid Geometry* which is being published by Ginn & Co. Something new and original may be expected.

Dr. G. A. Miller requests us to state that his position in Michigan University was an instructorship instead of a professorship. On account of his travels during the summer he failed to see the error until recently.

Prof. J. R. Baldwin has resigned his position as Professor of Mathematics in the Davenport Business College to accept a position at an increased salary in the Commercial Department in the High School of Davenport, Iowa.

Dr. Zerr notified us some time ago that his solution of Problem 43, Department of Geometry, is wrong. It does not follow from (3) and (4) that $y=z^2$. Professor Scheffer has pointed out the same error. We shall be pleased to publish a correct solution of this problem in the next issue of the MONTHLY.

We take pleasure in announcing that Drs. G. E. Fisher and I. J. Schwatt, of the University of Pennsylvania, have in press a translation of Durege's Theory of Functions of a Complex Variable, with special reference to Riemann surfaces. Durege's book is considered the standard text-book on the Theory of Functions. See comments on this book under Queries and Information.

The latest venture in the field of mathematical journalism in this country is the *American Mathematical Monthly*, edited by Profs. B. F. Finkel and J. M. Colaw, and issued monthly at $2. a year. The second volume is in progress and the double number for July and August, 1895, is at hand. The VISITOR heartily wishes the plucky editors the success they so richly deserve.

The Mathematical Visitor, 1895.

Prof. O. W. Anthony writes: "Allow me to congratulate you on the great success you are making of the MONTHLY. It is the best mathematical paper published for working mathematicians. I will send in my subscription for the coming year in a short time, and if there is any falling behind in financial matters, will be more than willing to bear my share." We are very thankful for Professor Anthony's kind words and generous offer of substantial support. We wish that the many hundreds of mathematicians of this country who are not now subscribers, would manifest the same spirit; they would then put their names upon our subscription list and contribute to the pages of the MONTHLY.

BOOKS AND PERIODICALS.

Elements of Geometry. By George C. Edwards, Ph. B., Associate Professor of Mathematics in the University of California. 8vo. cloth, 293 pp. Price, $1.10. New York: Macmillan & Co.

Some of the salient features of this new work are the concise and accurate statement of the definitions, the natural arrangement of the parts, the great generality of the demonstrations of many of the propositions, numerous interesting and valuable notes, and the development of method of attack in the solution of problems.

It is to be regretted that the author has omitted the subject of *Proportion*, giving as his reason that Proportion properly belongs to Algebra. While this is true, it is also true that many students begin the study of Geometry before they come to the study of Proportion in Algebra. But even if they have been drilled in the subject in Algebra, it has been my experience that the little time required for its discussion in Geometry is most helpful to even the brightest students, while its omission would prove very unsatisfactory to those who have not had it previously or who have had it several years previous to taking up Geometry.

The last chapter is devoted to the treatment of the Conic Sections. At the end of Plane Geometry and at the end of Solid Geometry there is given a large number of exercises designed to review the work preceding them, and thoroughly to establish method of attack in the mind of the student. Corollaries and scholia have been in large measure replaced by well chosen exercises. On pages 155—162 are thirty-nine diagrams to illustrate as many different demonstrations of the Pythagorean Proposition. The book is well written and the publishers have presented it for public favor in good style. B. F. F.

Plane and Spherical Trigonometry. By G. A. Wentworth, A. M., author of a series of text-books in Mathematics. Revised edition. 8vo. cloth and leather back, 192 pp. Price, $0.85. Boston and Chicago: Ginn & Co.

In preparing this work the aim has been to furnish just so much of Trigonometry as is actually taught in our best schools and colleges. Consequently all investigations that are important only for the special student have been omitted, except the development of functions in series. The principles have been unfolded with the utmost brevity consistent with simplicity and clearness, and interesting problems have been selected with a view of awakening a real love for the study. *Preface.*

The book is a good one and is most admirably adapted to the purpose for which it was prepared. B. F. F.

The Elements of Co-ordinate Geometry. By S. L. Loney, M. A., Late Fellow of Sidney Sussex College, Cambridge, Professor at the Royal Holloway College, and author of a Treatise on Elementary Dynamics, a Treatise on Plane Trigonometry, etc. 8vo. cloth, 416 pp. Price, $1.25. New York: Macmillan & Co.

This excellent book exemplifies the sound judgment and painstaking care which characterizes all of Professor Loney's mathematical works. He is putting himself in the front rank of mathematical writers of the present time, and his books will produce a healthful influence on the mathematicians of the future. We hope that the next book Mr. Loney writes will be a treatise on Spherical Trigonometry, thus making his Treatise on Trigonometry the most complete and admirable Treatise with which we are acquainted.
B. F. F.

The Mathematical Visitor. Edited and published by Artemas Martin, M. A., Ph. D., LL. D., United States Coast Survey, Washington, D. C. Quarto, 18 pp. Price, 50 cents. Issued annually.

The number for 1894 has just reached us. In it is published a number of different solutions of five different problems in Probability. The solutions are by Henry Heaton, G. B. M. Zerr, and the late Professor E. B. Seitz. Five excellent solutions of a difficult problem concerning the *curve of concealment* are also published. The solutions are by Dr. E. A. Bowser, Henry Heaton, Dr. Martin, the late Dr. J. E. Hendricks, and Charles H. Kummell. Two other interesting solutions of a problem are published. These are by J. F. W. Scheffer, and J. A. Pollard.

We regret very much that the ill health of Dr. Martin prevents his publishing the *Mathematical Visitor* and the *Mathematical Magazine* regularly. These two magazines are the type of excellence and beauty. B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year. Single number, 10 cents.

For complete and immediate revolution of transportation methods, involving a reduction of freight charges on grain from the West to New York of from 50 to 60 per cent, is what is predicted in the November *Cosmopolitan.* The plan proposes using light and inexpensive corrugated iron cylinders, hung on a slight rail supported on poles from a cross-arm—the whole system involving an expense of not more than fifteen hundred dollars a mile for construction. The rolling stock is equally simple and comparatively inexpensive. Continuous lines of cylinders, moving with no interval to speak of, would carry more grain in a day than a quadruple track railway. This would constitute a sort of grain-pipe line. *The Cosmopolitan* also points out the probable abolition of street cars before the coming horseless carriage, which can be operated by a boy on asphalt pavements at a total expense for labor, oil, and interest, of not more than one dollar a day. B. F. F.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Albert Shaw. Price, $2.50 per year. Single number, 25 cents. The Review of Reviews Co., New York City.

Foreign affairs naturally have more than usual prominence in the November *Review of Reviews.* In the "Progress of the World," the department of that periodical in which the editor rapidly reviews the events of the preceding month, the possibilities of war in the far East are pointedly set forth. Another theme suggested in the same connection is the progress of Christian missions in the Orient. The prospects of Japan and Russia as Eastern powers are tersely discussed. The editor also comments briefly on the relations of Russia and France, the Italian celebrations, the French victory in Madagascar, the Cuban situation, and British policy in Venezuela. Among home topics of the month, the coming elections, the condition of New York politics, the anti-prize-fight campaign in the Southwest, and the educational outlook are selected for treatment. B. F. F.

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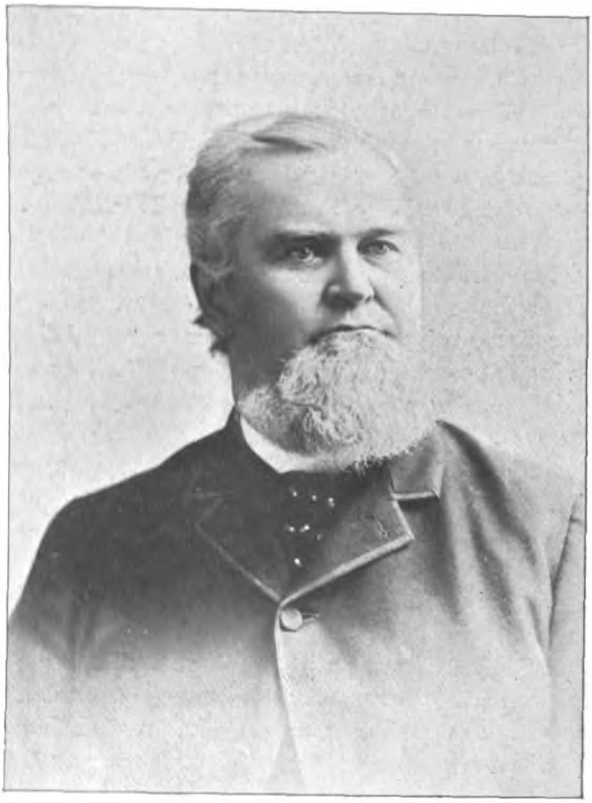


HUDSON A. WOOD.

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-Office at Springfield, Missouri, as Second-Class Mail Matter.

VOL. II. DECEMBER, 1895. No. 12.

BIOGRAPHY.

HUDSON A. WOOD.

BY F. P. MATZ, SC. D., PH. D., PROFESSOR OF MATHEMATICS AND ASTRONOMY IN IRVING COLLEGE, MECHANICSBURG, PENNSYLVANIA.

HUDSON A. WOOD, now Professor of Mathematics in the *Stevens School*, Hoboken, New Jersey, was born near Smyrna, New York, May 10, 1841. He is the middle one of a family of nine children; his brother, Professor DeVolson Wood, whose biography appeared in the September-October (1895) number of the MONTHLY, is the eldest. He was brought up on the farm, and early knew what hard work meant. He attributes his robust constitution to the vigorous exercise of his younger days. In the district school near his home, which he attended during the winter months, he acquired his early education. He evinced an unspeakable desire for study; and many a time, after a day's hard work, did he drop to sleep while poring over some book. At the age of fifteen, he spent his first winter away at school. When seventeen, he taught the district school adjoining his home; and, at the same time, he was initiated into the mysteries and pleasantries of boarding around. At the age of twenty he had taught a district school, a village school, and had completed the studies prescribed for the Freshman Class in Madison (now Colgate) University, at Hamilton, New York.

The year 1861, when Mr. Wood was twenty years of age, marks the beginning of the Civil War. A Company was raised at Hamilton, composed in part of students of the University. In this Company, afterward one of the Companies of the 61st Regiment of New York Volunteers, Mr. Wood enlisted. He was in the service nearly two years, and was engaged in six battles. His regiment took an active part in "The Seven-Days Battles" around Richmond, and sustained heavy losses. In the battle of Frazer's Farm, Mr. Wood had a ball shot through his coat, another through his haversack, and also received two slight flesh-wounds. More than one-half of the regiment fell in this desperate encounter at night-fall. He assisted in saving the colors of the regiment, for which he was promoted. At the battle of Fair Oaks, Mr. Wood stood within a few feet of General O. O. Howard, when the latter was wounded in the arm which afterward had to be amputated. In the battle of Malvern Hill, the regiment was hotly engaged for several hours; but owing to its protected position, the loss sustained by the regiment was not very severe. Soon after the battle of Malvern Hill, Mr. Wood was severely injured while working on the fortifications, and after lying in the hospital for over six months, and not recovering, he was discharged from the army.

Seven months after his return from the war, Mr. Wood entered the Literary Department of Michigan University. At the commencement exercises of the University, three years after his matriculation, he was among those chosen to deliver orations. Of Mr. Wood's oration, the Detroit *Tribune* spoke as follows: "His oration was one of the best of the day, both as to literary and elocutionary merit. Some portions were of unusual beauty, and the delivery was emphatic and impressive."

During his collegiate years, he spent the major portion of his time at Latin and Greek, as he found a thorough knowledge of these languages very difficult to acquire. For him, Mathematics always was "an easy study"—a *delightful* study; and for the Natural Sciences, he had (and still has) a peculiar *fondness.* On graduation he received the Degree of *Bachelor of Arts* (A. B.), in 1866; subsequently, the Degree of *Master of Arts* (A. M.); and last June, from New Windsor College, the Degree of *Doctor of Philosophy* (Ph. D.)

Mr. Wood was married to Miss Mary Hicks, near Rochester, New York, September 2, 1868; and he has two sons, 18 and 20 years of age, who are attending the Stevens Institute of Technology;

After graduation, Professor Wood was the Principal of an Academy near Philadelphia, Pa., for eight years, when he accepted the position of Vice Principal and Professor of Higher Mathematics and Astronomy in the *Keystone State Normal School* of Pennsylvania. During his connection with this School, Professor Wood edited the *Scientific Department*, and subsequently the *Mathematical Department*, in the NATIONAL EDUCATOR.

Among his pupils at this Institution, there was a rather slender, fair-faced, and affable *Pennsylvania-German* youth who had taken his Degree in the Pedagogical Course, during June of the same year in which Professor Wood, in August, entered upon his duties as Vice-Principal and Professor of Mathematics and Astronomy. This youth had returned to his *Alma Mater*, in order to take his Degree in the Scientific Course, two years later. He was the only student in the Scientific Course. Being an industrious student with a mathematico-scientific bent of mind, this youth soon had gained the friendship of Professor Wood. Like *father* and *son,* the professor and the youth enthusiastically studied the mathematical works of Loomis, Olney, Quinby, Courtenay, Bartlett, Todhunter—and even selections from the astronomical works of Chauvenet and Watson, for two long but profitable years. At the expiration of the second year, Professor Wood had the good fortune to see his youthful pupil (F. P. Matz) *passed* by the State Board of Examiners, and *graduated*, with "the highest distinction."

Afterwards Professor Wood held (for six years) a position in an educational institution in New York City, and subsequently was for three years the professor of Mathematics in the *Case School of Applied Science*, Cleveland, Ohio. From this last-named School, he was called to his present position, in 1888.

Dr. Wood is fond of Mathematics; and during the last twenty years, he has contributed articles and solutions of problems to *many* periodicals. Of late years, he has confined his attention more particularly to the works he is preparing for publication. His work, *Short Cuts and Curiosities in Mathematics*, is now passing through the press; and before the expiration of the current year, the American Book Company will have published his *Treatise on Plane and Spherical Trigonometry*. His *Perpetual Calendar*, good for ten centuries, has been pronounced the most unique calendar ever published. His article on *Method of Finding the Date of Easter*, has been highly commended. His *New Method of Extracting the Cube Root*, recently printed in the STEVENS INDICATOR, has been copied by numerous periodicals.

Dr. Wood has not confined his attention exclusively to Mathematics. He is well versed in the classics, well read in history, and an adept in geology. He is, also, an interesting speaker, and has delivered many public lectures illustrated with the stereopticon. His illustrated lectures on the Civil War are especially interesting.

As a teacher, Dr. Wood is earnest, untiring in his efforts, and patient to render assistance to those who acquire knowledge slowly. He is naturally a *leader*, and inspires his pupils with his own enthusiasm. He is the personification of kindness; but when he has to *drive*, he drives with an energy that is speedily satisfactory to those driven.

When Professor Wood left Cleveland, Ohio, in 1888, the following is an extract of what appeared in one of the large dailies of that city: "Professor H. A. Wood has become so well known in this city, and so highly esteemed by all who know him, that his contemplated change of residence will be felt as a great loss. He has made himself felt in the community as one always ready to do good. He has ever been formost in Sunday School work, mission enterprises, in church and social life, and in temperance and other reforms."

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from November Number.]

SCHOLION III, *in which is weighed the attempt of the Arab Nassaradin, and likewise the idea of the illustrious John Wallis upon the same affair.*

This endeavor of the Arab Nassaradin the already eulogised John Wallis has published in the Latin language with remarks added in opportune place.

However Nassaradin requires two things to be conceded to him in this affair.

The first is; that any two straight lines lying in the same plane, upon which ever-so-many other straight lines so strike, that they are always perpendicular to one indeed of these, but always cut the other at unequal angles, truly toward one part always under an acute angle, and toward the other always under an obtuse angle; that, I say, the above mentioned lines be supposed always more (as long as they do not mutually cut) to approach each other toward the side of those acute angles; and on the other hand always more to recede from one another toward the parts of the obtuse angles.

But I indeed, if nothing else impedes Nassaradin, willingly permit what he postulates; since just that, which with him remains undemonstrated can be recognized as most rigorously demonstrated by me in Cor. II. after P. III.

The other postulate of Nassaradin is the reciprocal of the first; that indeed the angle may always be acute toward those parts where the just mentioned perpendiculars are supposed to become always shorter; but obtuse toward the other parts where these perpendiculars are supposed to go out always longer. But here lurks an ambiguity.

For why (while from any one perpendicular prescribed as the first we proceed to the others) may not the angles of the consequent perpendiculars, on the same side acute, not become even greater, even to where one strikes upon a right angle, consequently upon such a perdendicular as is itself the common perpendicular to each of the aforesaid straights? And if indeed that happens, evanishes this subtle preparation of Nassaradin, by means of which ingeniously indeed, but with great labor he demonstrates the Euclidean postulate.

And yet if Nassaradin with a certain justice may determine to presume as if known 'per se' that persistence of acute angles on the same side: why can not also (I speak with Wallis) be assumed as if clear 'per se': *Two straights in the same plane converging* (upon which of course an other straight striking makes toward the same parts two angles less than two right angles, as suppose one right, and the other in whatever way acute) *finally meet, if produced?*

Nor in fact can it be objected, that this greater convergence toward one side can always subsist within a certain determinate limit, so that indeed a certain so much of distance always intervenes between these lines on this side, even if still one approaches always more nearly to the other.

That cannot, I say, be objected; since from this itself I will demonstrate, after P. XXV., the meeting at a finite distance of all such straights, in accordance with the Euclidean postulate.

Now I go over to the aforesaid John Wallis, who, as made a custom with so many great men, ancient as well as recent, and on the other hand from the obligation imposed on his Oxford professional chair, determined to undertake this same duty of demonstrating the oft mentioned postulate.

Now solely he assumes as if certain, what follows: namely that *to any given figure another similar of any magnitude is possible.*

And that this indeed may be presumed of any figure (although in his affair he assumes solely a rectilineal triangle) is well argued from the circle, which of course all admit can be described with any sized radius.

Further the acute man observes most cautiously it does not thwart this his presumption, that besides the equality of corresponding angles also the proportionality of all corresponding sides is required, in order that a rectilineal figure, for example a triangle, may be similar to another rectilineal triangle; though still the definition of proportion, and forthwith of similar figures are to be taken from the fifth, and the sixth book of Euclid: *For* (says he himself) *Euclid could have put each in front of book first.*

Hereafter, this standing (which nevertheless can be denied by any one, unless it is demonstrated) the famous man carries out his intent with really beautiful and ingenious effort.

But I am unwilling to fail in anything to the charge undertaken by me.

Therefore I assume two triangles, one ABC, and the other DEF (fig. 24) mutually equiangular. I do not say wholly similar; because I do not need the proportionality of the sides about the equal angles, nay nor any determinate measure of the sides themselves. Merely therefore I do not wish triangles mutually equilateral, since then the eighth of book first would alone suffice, without any assumption.

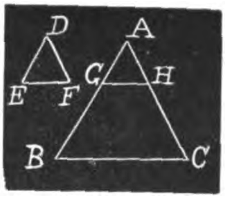


Fig. 24.

So let the angles at the points A, B, C, be equal to the angles at the points D, E, F; and let the side DE be less than the side AB; and in AB is assumed the portion AG equal to this DE, and likewise in AC the portion AH equal to this DF. But that DF must be less than AC I will make clear below. Then (GH joined) follows (from Eu. I. 4) the angles at the points E, and F will be equal to AGH, AHG. However since the just mentioned angles, together with the others BGH, CHG, are equal (Eu. I. 13) to four right angles; likewise will be equal to four right angles the angles at the points B, and C, together with these same angles BGH, CHG. Therefore the four angles of the quadrilateral $BGHC$ will be together equal to four right angles; and consequently (from P. XVI.) is established the hypothesis of right angle; and at the same time (from P. XIII.) the Euclidean postulate.

Moreover I have supposed the side DF, or AH assumed equal to it, to be less than the side AC. For if it were equal, and so the point H should fall upon the point C, then the angle BCA would be equal (by hypothesis) to the angle EFD, or GCA (which then it would become) a part to the whole; which is absurd.

But if it were greater, and so the join GH should cut BC itself in some point, now the external angle ACB would be from the hypothesis equal (against Eu. I. 16) to the internal and opposite angle (which then would become) AHG, or GHA.

Therefore I have rightly supposed the side DF of one triangle to be less than the side AC of the other triangle, in accordance with the hypothesis now established.

Wherefore from any two triangles mutually equiangular, but not also mutually equilateral, the Euclidean postulate is established.

Quod intendebatur.

[To be Continued.]

HISTORICAL SURVEY OF THE ATTEMPTS AT THE COMPUTATION AND CONSTRUCTION OF π.

By DAVID EUGENE SMITH, Ph. D., Professor of Mathematics in the Michigan State Normal School, Ypsilanti, Michigan.

[NOTE. The following article is translated (by permission) from Professor Klein's recent work, *Vortraege ueber ausgewaehlte Fragen der Elementargeometrie, ausgearbeitet von F. Taegert*, Leipzig, Teubner, 1895. The work can not be too highly commended to teachers, since it is one of those exceedingly rare treatises in which a master of modern mathematics has treated elementary subjects from his high point of view.
Michigan State Normal School, December, 1895.]

Later in this work it will be proved that the number π belongs to that class of numbers known as transcendent, whose existence was shown in the preceding chapter. This fact was first proved by Lindemann in 1882, and a problem was thereupon settled which, so far as our information extends, has occupied the attention of mathematicians for 4000 years, namely, that of the quadrature of the circle.

It is evident that if the number π is not algebraic it cannot be constructed by means of the compasses and ruler. Hence the quadrature of the circle is, in the sense understood by the ancients, impossible. It is of greatest interest to follow the fortunes of this problem in the various epochs of Science, as ever new attempts were made to find a solution by means of the ruler and the compasses, and to see how these necessarily fruitless attempts nevertheless worked for advancement in the manifold realm of mathematics.

The following brief historical survey is based upon Rudio's excellent treatise, *Archimedes, Huygens, Lambert, Legendre; Vier Albandlungen ueber die Kreismessung, Leipzig*, 1892. In this work are given in German translation the contributions of the writers named. Even though the presentation of the matter is remote from the more modern methods here discussed,* nevertheless it includes many very interesting details which are of especial value in elementary teaching.

1. Among the attempts to determine the ratio of the diameter to the circumference we may first distinguish the empirical stage in which it was sought to attain the desired end through measuring or estimating. The oldest known mathematical work, the Rhind Papyrus (c. 2000 B. C.) contains the problem in the well-known form, to transform a circle into a square of equal area. The writer of the papyrus, Ahmes, lays down the following rule: Cut off $\frac{1}{9}$ of a diameter and construct a square on the remainder; this has the same area as the circle. The value of π thus obtained is $(\frac{16}{9})^2=3.16\ldots\ldots$, not very inexact. Still farther from the correct value is that of $\pi=3$ which is found in the Bible. (I Kings, 7:23, and II Chron. 4:2.)

2. The Greeks raised themselves above this impirical standpoint, and especially Archimedes, who in his work *Κύκλου μέτρησις* computes the area of the circle by the help of inscribed and circumscribed polygons, as is still done in the schools. His method remained in use until the invention of the differential calculus, and was extended and made practically usable especially by Huygens (†1654) in his work *De circuli magnitudine inventa.*

As in the case of the duplication of the cube and the trisection of an angle the Greeks then sought to attain the quadrature of the circle by the help of higher curves.

We may, for example, consider the curve, $y=\text{arc}\sin x$ [usually written in English $y=\sin^{-1}x$; the Continental form will be followed in this translation] which represents the curve of sines placed vertically. Geometrically, π appears as a special ordinate of this curve, analytically as a special value of our transcendent function. Apparatus which describes transcendent curves we will call transcendent apparatus. A piece of transcendent apparatus which draws the curve of sines gives us a real construction for π. The curve $y=\text{arc}\sin x$ we designate now-a-days as an *integral curve*, because it can be defined by means of the integral of an algebraic function: $y=\int\frac{dx}{\sqrt{1-x^2}}$. The ancients called such a curve a Quadratrix or *τετραγωνίζουσα*. The best known of these is the Quadratrix of Dinostratus (c. 350 B. C.) which, however, had been already constructed by Hippias of Elis (c. 420 B. C.) for the trisection of an angle. It may be geometrically defined as follows: On the line OB and the arc AB two points, M and L, move

*In a note to the translator Professor Klein says: "This remark concerning Rudio's work is not happily expressed. The meaning is not that modern researches, so far as then carried, are not given in the work, but they are not deduced."

with uniform velocity. They start at the same time from O and A, respectively, and they reach B at the same time. If OL is drawn, and through M the parallel to OA which meets OL at P, then P is a point of the Quadratrix. From this definition it follows that y and θ are proportional. Further, since for $y=1$, $\theta=\frac{\pi}{2}$, we have $\theta=\frac{\pi}{2}y$, and from $\theta=\text{arc tan}\frac{y}{x}$ the equation of the curve becomes $\frac{y}{x}=\tan\frac{\pi}{2}y$. The point in which the line cuts the x-axis will be found from $x=\frac{y}{\tan\frac{\pi}{2}y}$ if y becomes 0. Since for small values the tangent equals its argument, it follows that $x=\frac{2}{\pi}$. Hence the radius of the circle is the mean proportional between the quadrant of the circle and the abscissa of the point of intersection of the Quadratrix with the x-axis. The Quadratrix can, therefore, be used in the rectification problem, and hence for the quadrature of the circle. Fundamentally, however, the curve is only a geometric formulation of the rectification problem, that is so long as no apparatus is given by which it can be described by a continuous line.

3. The rise of modern analysis occurs in the period from 1670 to 1770, a period characterized by the names of Leibniz, Newton, and Euler. In the midst of so many great discoveries following closely on one another, it is natural that strict criticism took a somewhat backward step. Among these discoveries is one of especial concern to us, the development of the theory of series. Especially for π were a great number of approximations brought forward, of which we may mention only the so-called Leibniz series (which, however, was known before Leibniz): $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots$ Furthermore this period brings the discovery of the connection between e and π. The number e and the natural logarithms and with them the exponential function are first found *in embryo* in the works of Napier (1614). This number seemed at first to have no relation to the circular functions and to the number π, until Euler had the courage to attack the problem by means of imaginary exponents. In this way he reached the celebrated formula $e^{ix}=\cos x+i\sin x$, which for $x=\pi$ becomes $e^{i\pi}=-1$. This formula is without doubt one of the most notable of all mathematics. With it are connected the modern proofs of the transcendence of π since they first show the transcendence of e.

4. After 1770 criticism again took the upper hand. In 1770 appeared Lambert's work, *Vorläufige Kentnisse für die, so die Quadratur des Cirkuls suchen.* He treated there and elsewhere the irrationality of π. In 1794 Legendre showed conclusively in his *Éléments de Géométrie* that π and π^2 are irrational numbers.

5. But it was not until a hundred years later than this that modern research began. The starting point of this research is the work of Hermite, *Sur la fonction exponentielle* (*Compt. Rend.* 1873, published separately in 1874). In this

is proved the transcendence of e. Closely following Hermite came the same proof for π by Lindemann in a dissertation *Ueber die Zahl* π (*Math. Ann.* 20, 1882. See also the proceedings of the Berlin and Paris academies). With this the matter was now for the first time settled, nevertheless the treatment given by Hermite and Lindemann is very complicated.

The first simplification was given by Weierstrass in the *Berliner Berichte* in 1885. The above mentioned works Bachman embodied in his text-book, *Vorlesungen ueber die Natur der Irrationalzahlen*, 1892.

The spring of 1893 brought, however, new and very important simplifications. In the first rank should be named the developments of Hilbert in the *Göttinger Nachrichten.* Hilbert's proof is not wholly elementary; it contains still a remnant of Hermite's course of reasoning in the integral

$$\int_0^\infty z^\rho e^{-z} dz = \rho\,!.$$

But Herwitz and Gordan soon after showed that this transcendental part might be eliminated. (*Göttinger Nachrichten* and *Comptes Rendus* respectively; all three dissertations are reproduced in the *Math. Annalen*, Bd. 43, either literally or somewhat extended). So the matter has now become so elementary that it is generally available.

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from November Number.]

The Construction of Non-Primitive Groups with Two Systems of Non-Primitivity.

Let the degree of the required non-primitive group be $2n$, and consider the $(n\,!)^2$ substitutions

$$(a_1 a_2 \ldots\ldots a_n)\text{all}(b_1 b_2 \ldots\ldots b_n)\text{all } a_1 b_1 . a_2 b_2 \ldots\ldots a_n b_n$$

and also the group of order $2(n\,!)^2$

$$(a_1 a_2 \ldots\ldots a_n)\text{all}(b_1 b_2 \ldots\ldots b_n)\text{all}(a_1 b_1 . a_2 b_2 \ldots\ldots a_n b_n).$$

The latter is clearly a non-primitive group of degree $2n$ and the former are the substitutions of this group which interchange the systems. It is easily seen that G_1 can have no larger value than it has in the above non-primitive group, and that every G_1 for other non-primitive groups may be regarded as a subgroup of this G_1. From this it follows that the first set of substitutions includes all the substitutions which can be used with any G_1 to form a non-primitive group, for if there were such a substitution s_x which is not in the first set then we would obtain more than $(n!)^2$ different substitutions which transform

$$(a_1 a_2 \ldots\ldots a_n)\text{all}(b_1 b_2 \ldots\ldots b_n)\text{all}$$

into itself without interchanging the systems by multiplying one substitution of this set into the entire set increased by s_x. Hence all the substitutions which can be used to interchange the systems are found in the first set. In a similar way we can show that the number of the snbstitutions which interchange the systems must *always* be equal to the order of G_1. Hence if in any non-primitive group we represent the substitution which interchange the systems by G_2 and the non-primitive group by G we have

$$G = G_1 + G_2$$

where G_1 and G_2 contain the same number of substitutions and G is a subgroup of

$$(a_1 a_2 \ldots\ldots a_n)\text{all}(b_1 b_2 \ldots\ldots b_n)\text{all}(a_1 b_1 . a_2 b_2 \ldots\ldots a_n b_n).$$

Suppose any G_1 constructed by combining a transitive* subgroup of $(a_1 a_2 \ldots\ldots a_n)$all with a conjugate subgroup of $(b_1 b_2 \ldots\ldots b_n)$all and suppose s_y to have the following properties :

(1) its square is found in G_1;
(2) it transforms G_1 into itself ;
(3) it interchanges the systems of G_1.

Then will all of the substitutions

$$G_1 s_y = G_2$$

*We shall henceforth assume that the systems of non-primitivity are the transitive constituents of G_1. We proved above that this can always be done but we did not prove that it is possible to regard intransitive constituents of G_1 as systems. That this may be done is proved by the following instance in which

$$G_1 = (ab.cd.ef.gh.ij.kl)$$

and the systems are either a,b; c,d; e,f; g,h; i,j; and k,l or a,b,c,d; e,f,g,h; i,j,k,l. Letting the letters A, B, etc., stand for the first systems and A', B', C' for the second we may write the group as follows:

1 ab.cd.ef.gh.ij.kl	1
aei.bfj.cgk.dhl afibcj.chkdgl	ABC.DEF A'B'C'
aie.bjf.ckg.dlh ajebif.clgdkh	ACB.DFE A'C'B'
ac.bd.ek.fl.gi.hj ad.bc.el.fk.gj.hi	AD.BF.CE B'C'
ag.bh.ce.df.ik.jl ah.bg.cf.de.il.jk	AE.BD.CF A'B'
ak.bl.ci.dj.eg.fh al.bk.cj.di.eh.fg	AF.BE.CD A'C'

If we consider the six systems they are the transitive constituents of G_1 but if we consider only the three systems they are intransitive constituents.

have these properties and it can be easily seen that G_1+G_2 constitute a non-primitive group. Hence it follows that it is only necessary to find one substitution which possesses the three properties named above in order to obtain a G_2 corresponding to a given G_1.

To fix these ideas we proceed to find all the non-primitive groups whose degree does not exceed six. Since n must be the degree of some group it follows that $2n$ cannot be less than four.

Non-Primitive Groups of Degree Four.

G_1 must be either $(ac.bd)$ or $(ac)(bd)$. G_2 is found in $(ac)(bd)ab.cd=ab.cd$, $abcd$, $adcb$, $ad.bc$. If $G_1=(ac.bd)$ we see at once that $ab.cd$ and $abcd$ satisfy the three required conditions. In the first case $G_2=ab.cd$, $ad.bc$ and in the second case it equals $abcd$, $adcb$. Hence the given G_1 leads to the following two non-primitive groups of degree and order four : (A transitive group is called *regular* when its degree is equal to its order.)

$$(abcd)_4,\ (abcd)\text{cyc}.$$

If $G_1=(ac)(bd)$ we see again directly that $ab.cd$ satisfies the three required conditions, as we found in the general case. The corresponding G_2 includes all the possible substitutions. We obtain therefore only one non-primitive group with this G_1, viz :

$$(abcd)_8$$

Hence there are *three* non-primitive groups of degree four. The other two transitive groups of degree four are multiply transitive and therefore primitive.

Non-Primitive Groups of Degree Six with Two Systems of Non-Primitivity.

G_1 must be one of the following five groups :

$$(abc)\text{all}(def)\text{all},\ \{(abc)\text{all}(def)\text{all}\}\text{pos},\ (abc.def)\text{all}$$
$$(abc)\text{cyc}(def)\text{cyc},\ (abc.def)\text{cyc}.$$

G_2 is found in

$$(abc)\text{all}(def)\text{all}\ ad.be.cf.$$

(*a*) If $G_1=(abc)\text{all}(def)\text{all}$, G_2 will include all the possible substitutions and we obtain one group of order 72, viz :

$$(1)\qquad (abc)\text{all}(def)\text{all}(ad.be.cf).$$

(*b*) If $G_1=\{(abc)\text{all}(def)\text{all}\}\text{pos}$ the two substitutions $ad.be.cf$ and $aebd.cf$ satisfy the three conditions and we thus obtain one G_2 which contains only negative substitutions and another which contains only positive substitutions. The two resulting groups are

(2) $\{(abc)\text{all}(def)\text{all}\}\,\text{pos}(ad.be.cf)=(abcdef)_{36},$*

(3) $\{(abc)\text{all}(def)\text{all}\}\,\text{pos}(aebd.cf)=(abcdef)_{36}.$

(*c*) If $G_1=(abc.def)\text{all}$ $ad.be.cf$ satisfies the three necessary conditions. We thus obtain

(4) $$(abc.def)\text{all}(ad.be.cf)=(abcdef)_{12}.$$

No substitutions except those in the above G_2 can transform $(abc.def)$all into itself and interchange the systems, because no two substitutions of (abc)all transform (abc)all in the same way. Hence there is only one G_2 for the given G_1.

(*d*) If $G_1=(abc)\text{cyc}(def)\text{cyc}$, then the square of only half of the substitutions in which G_2 is found are contained in this G_1. Hence two G_2's are possible, viz:

$$G_1\ ad.be.cf \text{ and } G_1\ ab.de.ad.be.cf=G_1\ ae.bd.cf.$$

ab transforms G_1 into itself and one of these G_2's into the other so that there is really only the following non-primitive group with the given G_1†:

(5) $$(abc)\text{cyc}(def)\text{cyc}(ad.be.cf).$$

(*e*) Finally if $G_1=(abc.def)\text{cyc}$ we obtain two G_2's and hence the following groups:

(6) $$(abc.def)\text{cyc}(ad.be.cf)=(abcdef)\text{cyc}$$

(7) $$(abc.def)\text{cyc}(ae.bd.cf)=(abcdef)_6$$

The first one of these two will be found in three conjugate forms if we use all the possible G_2's. We have now examined all the possible G_1's and found seven non-primitive groups of degree six which contain two systems of non-primitivity.

[To be Continued.]

*This group is not found in Professor Cayley's list, *Quarterly Journal of Mathematics*, Vol. 25. pp. 71—79. It is found in Professor Cole's supplementary list, *Bulletin of the New York Mathematical Society*, May, 1893.

†It has been proved that whenever G_1 is the product of two groups then there is really only one G_2 for the given G_1. We shall give a proof of this theorem later.

THE RECTIFICATION OF THE CASSINIAN OVAL BY MEANS OF ELLIPTIC FUNCTIONS.

By F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

[Continued from September-October Number.]

IV. Typical of the Cassinian Oval, we have by the *Method of Complex Variables* the following equations:

$$x=\beta\sqrt{(1+\alpha t)},=\beta\sqrt{[1+\alpha(\cos\theta+i\sin\theta)]}\ldots\ldots(A);$$

$$y=\beta\sqrt{(1+\alpha/t)},=\beta\sqrt{[1+\alpha(\cos\theta-i\sin\theta)]}\ldots\ldots(B).$$

$$\therefore\ dx/d\theta=\tfrac{1}{2}i^2\alpha\beta(\cos\theta-i\sin\theta)/\sqrt{[1+\alpha(\cos\theta+i\sin\theta)]}\ldots\ldots(A');$$

$$dy/d\theta=\tfrac{1}{2}i^2\alpha\beta(\cos\theta+i\sin\theta)/\sqrt{[1+\alpha(\cos\theta-i\sin\theta)]}\ldots\ldots(B').$$

$$\therefore\left(\frac{d\mathbf{P}_1}{d\theta}\right)^2=\frac{i^4\alpha^2\beta^2(\cos\theta-i\sin\theta)(\cos\theta+i\sin\theta)}{4\sqrt{\{[1+\alpha(\cos\theta+i\sin\theta)]\times[1+\alpha(\cos\theta-i\sin\theta)]\}}}\ldots\ldots(C).$$

$$\therefore\ \mathbf{P}=\alpha\beta\int_0^{2\pi}\frac{d\theta}{[1+\alpha^2+2\alpha\cos\theta]^{\frac{1}{2}}}=\frac{2\alpha\beta}{[1+\alpha^2]^{\frac{1}{2}}}\int_0^{\pi}\frac{d\theta}{[1+M\cos\theta]^{\frac{1}{2}}}\ \ldots\ldots(D).$$

[From (A) and (B), after differentiating with respect to t, we have

$$dx/dt=\tfrac{1}{2}\alpha\beta/x=\tfrac{1}{2}\alpha\beta/\sqrt{(1+\alpha t)},$$

$$dy/dt=-\tfrac{1}{2}\alpha\beta/t^2y=-\tfrac{1}{2}\alpha\beta/t^2\sqrt{(1+\alpha/t)}.$$

$$\therefore\ (d\mathbf{P}_1)^2=\left(\frac{-\alpha^2\beta^2}{4\sqrt{[(1+\alpha t)(1+\alpha/t)]}}\right)\left(\frac{dt}{t}\right)^2\ldots\ldots(E).$$

Differentiating under the assumption that $t=\cos\theta+i\sin\theta$, etc.,

$$\frac{dt}{t}=\left(\frac{i\cos\theta-\sin\theta}{\cos\theta+i\sin\theta}\right)d\theta=i\left(\frac{\cos\theta+i\sin\theta}{\cos\theta+i\sin\theta}\right)d\theta=id\theta.$$

$$\therefore\ \mathbf{P}_1=\tfrac{1}{2}i\alpha\beta\int\frac{dt}{t[1+\alpha^2+\alpha(t+1/t)]^{\frac{1}{2}}}=\tfrac{1}{2}\alpha\beta\int\frac{d\theta}{[1+\alpha^2+2\alpha\cos\theta]^{\frac{1}{2}}};$$

$$\text{and } \mathbf{P}=\tfrac{1}{2}\alpha\beta\int_0^{4\pi}\frac{d\theta}{[1+\alpha^2+2\alpha\cos\theta]^{\frac{1}{2}}}=\frac{2\alpha\beta}{[1+\alpha^2]^{\frac{1}{2}}}\int_0^{\pi}\frac{d\theta}{[1+M\cos\theta]^{\frac{1}{2}}}.\Big]$$

Since in the Cassinian Oval under consideration, $\alpha=\frac{5}{4}$ and $\beta=2$, we have $M=\frac{40}{41}$; that is, (D) is *identical* with (4) on page 265 of the September-October MONTHLY. Slowly converging series may be obtained by transforming under the hypothesis that $\theta=(90-\phi)$, or under the hypothesis that $\theta=(90+\phi)$.

V. The assumption of (2) from page 264 of the MONTHLY specified gives

$$\mathbf{P}=m^2\int_b^a\frac{r^2dr}{\sqrt{\{[(m^2+c^2)^2-r^4]\times[r^4-(m^2-c^2)^2]\}}}\ldots\ldots(2).$$

Let $(m^2-c^2)/(m^2+c^2)=e^2$, and $r^2=(m^2+c^2)x^2\ldots\ldots(k)$;

$$\text{then } \mathbf{P}=\frac{4m^2}{\sqrt{(m^2+c^2)}}\int_e^1\frac{2(m^2+c^2)^2x^2dx}{\sqrt{\{[(m^2+c^2)^2(1-x^4)]\times[(m^2+c^2)^2(x^4-e^4)]\}}}\ldots(\delta),$$

$$\text{or } \mathbf{P}=\frac{4m^2}{\sqrt{(m^2+c^2)}}\int_e^1\frac{2x^2dx}{\sqrt{[(1-x^4)(x^4-e^4)]}}\ldots\ldots(F),$$

$$=\frac{4m^2}{\sqrt{(m^2+c^2)}}\Big[\int_e^1\frac{(x^2-e)dx}{\sqrt{[(1-x^4)(x^4-e^4)]}}+\int_e^1\frac{(x^2+e)dx}{\sqrt{[(1-x^4)(x^4-e^4)]}}\Big]\ldots\ldots(G),$$

$$=\sqrt{\Big\{\Big(\frac{16m^4}{m^2+c^2}\Big)\Big(\frac{1}{2(1+e^2)}\Big)\Big\}}\Big[cn^{-1}\Big(\frac{x+e/x}{1+e},\ \frac{1+e}{\sqrt{[2(1+e^2)]}}\Big)$$

$$+\,cn^{-1}\Big(\frac{x-e/x}{1-e},\ \frac{1-e}{\sqrt{[2(1+e^2)]}}\Big)\Big]_1^e=\sqrt{\Big\{\Big(\frac{16m^4}{m^2+c^2}\Big)\Big(\frac{1}{2(1+e^2)}\Big)\Big\}}\Big[\{cn^{-1}(+1)$$

$$-cn^{-1}(+1)\}+\{cn^{-1}(-1)-cn^{-1}(+1)\}\Big]\ldots\ldots(H);$$

that is, the *first* indicated integral in (G) *has vanished*. The expression for the perimeter of the *Cassinian Oval*, therefore, becomes

$$\mathbf{P}=2\pi m\Big\{1+\Sigma\Big(\frac{1.3.5.7\ldots\ldots(2n-1)}{2.4.6.8\ldots\ldots\ldots.2n}\Big)^2\Big[\sqrt{\Big(1+\frac{c^2}{m^2}\Big)}-\sqrt{\Big(1-\frac{c^2}{m^2}\Big)}\Big]^{2n}\Big\},$$

which is a *complete* elliptic integral of the *first* order.

For the perimeter of the *Bernoullian Lemniscate*, we have $m=c$; that is, symmetrically expressed,

$$\mathbf{P}'=2\pi c\Big\{1+\Sigma\Big(\frac{1.3.5.7\ldots\ldots(2n-1)}{2.4.6.8\ldots\ldots\ldots.2n}\Big)^2\Big[\sqrt{\Big(1+\frac{c^2}{c^2}\Big)}-\sqrt{\Big(1-\frac{c^2}{c^2}\Big)}\Big]^{2n}\Big\}.$$

For the perimeter of the two *Ovaliform Figures*, we have $m\angle c$; that is, similarly expressed,

$$P' \sqrt{2\pi c}\left(\frac{m}{c}\right)^2 \left\{1+\Sigma\left(\frac{1.3.5.7\ \ldots\ldots(2n-1)}{2.4.6.8\ldots\ldots\ldots\ldots2n}\right)^2\left[\sqrt{\left(1+\frac{m^2}{c^2}\right)}-\sqrt{\left(1-\frac{m^2}{c^2}\right)}\right]^{2n}\right\}.$$

[Concluded.]

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

54. Proposed by D. P. WAGONER, A. B., Principal of the School of Language, Westerville, Ohio.

A man bought a farm for $6000 and agreed to pay for it in four equal annual installments, at 6 per cent. annual interest compounded every instant. Required the annual payment. *B. F. Burleson.*

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana, College, Texarkana, Arkansas-Texas; P. S. BERG, Larimore, North Dakota; and J. SCHEFFER, A. M., Hagerstown, Maryland.

Let a=$6000, $r=.06$, x=annual payment, and $m=4$=number of equal annual payments. If the interest is compounded n times a year, we have the present value of the first installment$=x(1+\frac{r}{n})^{-n}=xe^{-r}$ when n is infinite; of the second, $=xe^{-2r}$; of the third, $=xe^{-3r}$; of the mth, $=xe^{-mr}$; where e= Naperian base.

(See Todhunter's Differential Calculus, page 136).

$$\therefore\ a=x\left(\frac{1}{e^r}+\frac{1}{e^{2r}}+\frac{1}{e^r}+\ldots\ldots+\frac{1}{e^{mr}}\right)=\frac{x}{e^{mr}}\left(\frac{e^{mr}-1}{e^r-1}\right).$$

$$\therefore\ x=\frac{a\ e^{mr}(e^r-1)}{e^{mr}-1}=\frac{a(e^r-1)}{1-e^{-mr}}=\frac{a(e^r-1)}{1-e^{-4r}}$$

$$\therefore\ x=\$6000\left(\frac{e^{.06}-1}{1-e^{-.24}}\right)=\$1738.269.$$

II. Solution by B. F. BURLESON, Oneida Castle, New York.

The amount of P in n years at $r=6\%$ when the interest is compounded q times a year is evidently

$$A=P(1+\frac{r}{q})^{nq}\ldots\ldots(1).$$

Expanding the right hand member in (1) by the binomial theorem, we have

$$A=P(1+nq.\frac{r}{q}+\frac{nq(nq-1)}{2\,!}.\frac{n^2}{q^2}+\ldots\ldots(2).$$

When $q=\infty$, equation (2) becomes

$$A=P(1+nr+\frac{n^2r^2}{2\,!}+\frac{n^3r^3}{3\,!}+\text{etc.,})=,$$

by the exponential theorem, Pe^{nr} .....(3).

Whence by taking logarithms in (3) and changing to the common system by multiplying by its modulus we have in inverse functions

$$A=P\log(1.4342944nr)=106.1836,$$

when $n=1$, $r=.06$, and $P=100$.

Having determined the rate in equivalent annual compound rate, the required annual payment is determined as follows:

$1\div1.061836=.941765=P$

$1\div1.061836^2=.886913=P'$

$1\div1.061836^3=.8352716=P''$

$1\div1.061836^4=.7866294=P'''$.

Now $\$6000\div(P+P'+P''+P''')=\$1738.834$.

This problem was also solved by *B. F. YANNEY.*

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

50. Proposed by LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago.

Given $b=a\sqrt{-1}.\tan\frac{m\pi}{n}$, m being an arbitrary integer, find the simplest *algebraic* relation between a and b.

Solution by the PROPOSER.

From $b=ai\tan\frac{m\pi}{n}$, where $i=\sqrt{-1}$, we derive $\frac{a+b}{a-b}=\frac{1+i\tan\frac{m\pi}{n}}{1-i\tan\frac{m\pi}{n}}$

$=\frac{1-\tan^2\frac{m\pi}{n}+2i\tan\frac{m\pi}{n}}{1+\tan^2\frac{m\pi}{n}}=\cos^2\frac{m\pi}{n}-\sin^2\frac{m\pi}{n}+2i\sin\frac{m\pi}{n}.\cos\frac{m\pi}{n}=\cos\frac{2m\pi}{n}$

$+i\sin\frac{2m\pi}{n}$, the n^{th} power of which, by De Moivre's Theorem, equals 1.

$\therefore (a+b)^n=(a-b)^n$.

Also solved by *F. P. MATZ.*

51. Proposed by J. W. NICHOLSON, LL. D., President, and Professor of Mathematics, Louisiana State University and A. and M. College, Baton Rouge, Louisiana.

Solve the equation $x^5+5mx^3+5m^2x+n=0$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let $x=y+z$. The equation then easily reduces to $y^5+z^5+5(yz+m)\{y^3+z^3+(2yz+m)(y+z)\}+n=0$. Now x may be decomposed into two parts, y and z, in an infinite variety of ways, and we may, therefore, suppose y and z are such as to satisfy the condition $yz+m=0$. This gives $yz=-m$, $y^5+z^5=-n$. Let $y^5=t_1$, $z^5=t_2$, then we have $t_1t_2=-m^5$, $t_1+t_1=-n$. $\therefore t_1$ and t_2 are the roots of the equation $t^2+nt-m^5=0\ldots\ldots(1)$. $\therefore x=y+z=\sqrt[5]{t_1}+\sqrt[5]{t_2}$.

Case I. When m is positive.

Let $t=u\sqrt{m^5}$, then (1) becomes $u^2+\frac{n}{\sqrt{m^5}}u-1=0$; but $\tan^2\frac{1}{2}\theta+2\cot\theta\tan\frac{1}{2}\theta-1=0$. $\therefore 2\cot\theta=n/\sqrt{m^5}$, or $\tan\theta=2\sqrt{m^5}/n$. $\therefore t_1=\sqrt{m^5}\tan\frac{1}{2}\theta$, $t_2=-\sqrt{m^5}\cot\frac{1}{2}\theta$. $\therefore x=\sqrt{m}(\sqrt[5]{\tan\frac{1}{2}\theta}-\sqrt[5]{\cot\frac{1}{2}\theta})$, where $\theta<90^\circ$. Four of the five roots of $\sqrt[5]{\tan\frac{1}{2}\theta}$ are imaginary. Let $\tan\frac{1}{2}\phi=r=$the real value of $\sqrt[5]{\tan\frac{1}{2}\theta}$, and let a_1, a_2, a_3, a_4 represent the four imaginary roots of unity

$\frac{\sqrt{5}-1+\sqrt{-10-2\sqrt{5}}}{4}$, $\frac{\sqrt{5}-1-\sqrt{-10-2\sqrt{5}}}{4}$, $\frac{-\sqrt{5}-1-\sqrt{-10+2\sqrt{5}}}{4}$,

$\frac{-\sqrt{5}-1+\sqrt{-10+2\sqrt{5}}}{4}$, respectively.

Then $x_1=\sqrt{m}(r-\frac{1}{r})$, $x_2=\sqrt{m}(ra_1-\frac{1}{ra_1})$, $x_3=\sqrt{m}(ra_2-\frac{1}{ra_2})$, $x_4=\sqrt{m}(ra_3=\frac{1}{ra_3})$, $x_5=\sqrt{m}(ra_4-\frac{1}{ra_4})$. Substituting the values of r, a_1, a_2, a_3, a_4, we get $x_1=-2\sqrt{m}\cot\phi$,

$x_2=-\frac{1}{2}\sqrt{m}\{(\sqrt{5}-1)\cot\phi-\sqrt{-10-2\sqrt{5}}\operatorname{cosec}\phi\}$,

$x_3=-\frac{1}{2}\sqrt{m}\{(\sqrt{5}-1)\cot\phi+\sqrt{-10-2\sqrt{5}}\operatorname{cosec}\phi\}$,

$x_4=\frac{1}{2}\sqrt{m}\{(\sqrt{5}+1)\cot\phi+\sqrt{-10+2\sqrt{5}}\operatorname{cosec}\phi\}$,

$x_5=\frac{1}{2}\sqrt{m}\{(\sqrt{5}+1)\cot\phi-\sqrt{-10+2\sqrt{5}}\operatorname{cosec}\phi\}$.

Case II. When m is negative and $-4m^5<n^2$.

Then (1) becomes $t^2+nt+(-m^5)=0$, or $u^2+\dfrac{n}{\sqrt{-m_5}}u+1=0$; but $\tan^2\frac{1}{2}\theta-2\operatorname{cosec}\theta\tan\frac{1}{2}\theta+1=0$. $\therefore -2\operatorname{cosec}\theta=n/\sqrt{-m^5}$, or $\sin\theta=-2\sqrt{-m^5}/n$.

$\therefore t_1=\sqrt{-m^5}\tan\frac{1}{2}\theta$, $t_2=\sqrt{-m^5}\cot\frac{1}{2}\theta$. $\therefore x=\sqrt{-m}(\sqrt[5]{\tan\frac{1}{2}\theta}+\sqrt[5]{\cot\frac{1}{2}\theta})$.

By a process similar to that in Case I, we get,

$x_1=\sqrt{-m}(r+\frac{1}{r})=2\sqrt{-m}\operatorname{cosec}\phi$,

$x_2=\sqrt{-m}(ra_1+\frac{1}{ra_1})=\frac{1}{2}\sqrt{-m}\{(\sqrt{5}-1)\operatorname{cosec}\phi-\sqrt{-10-2\sqrt{5}}\cot\phi\}$,

$x_3=\sqrt{-m}(ra_2+\frac{1}{ra_2})=\frac{1}{2}\sqrt{-m}\{(\sqrt{5}-1)\operatorname{cosec}\phi+\sqrt{-10-2\sqrt{5}}\cot\phi\}$,

$x_4=\sqrt{-m}(ra_3+\frac{1}{ra_3})=-\frac{1}{2}\sqrt{-m}\{(\sqrt{5}+1)\operatorname{cosec}\phi-\sqrt{-10+2\sqrt{5}}\cot\phi\}$,

$x_5=\sqrt{-m}(ra_4+\frac{1}{ra_4})=-\frac{1}{2}\sqrt{-m}\{(\sqrt{5}+1)\operatorname{cosec}\phi+\sqrt{-10+2\sqrt{5}}\cot\phi\}$.

Case III. When m is negative and $-4m^5<n^2$.

In this case the preceding method fails. Let $x=ku$, then $x^5+5mx^3+5m^2x+n=0$ becomes $u^5+\dfrac{5mu^3}{k^2}+\dfrac{5m^2u}{k^4}+\dfrac{n}{k^5}=0$; also $\cos^5\theta-\frac{5}{4}\cos^3\theta+\frac{5}{16}\cos\theta-\frac{1}{16}\cos5\theta=0$. Let $u=\cos\theta$, then $5m/k^2=-5/4$.

$\therefore k=2\sqrt{-m}$, $-\frac{1}{16}\cos5\theta=\dfrac{n}{k^5}=\dfrac{n}{32m^2\sqrt{-m}}$, $\therefore \cos5\theta=-\dfrac{n}{2m^2\sqrt{-m}}$.

Since $-m$ is positive, this gives five real values of θ, to be taken $<90°$.

$\therefore x_1=2\sqrt{-m}\cos\theta$, $x_2=2\sqrt{-m}\cos(\frac{2\pi}{5}-\theta)$, $x_3=2\sqrt{-m}\cos(\frac{2\pi}{5}+\theta)$, $x_4=2\sqrt{-m}\cos(\frac{4\pi}{5}-\theta)$, $x_5=2\sqrt{-m}\cos(\frac{4\pi}{5}+\theta)$.

II. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

Let $x=z-\dfrac{m}{z}$, and the equation reduces to $z^{10}+nz^5=m^5$.

Whence $z=\sqrt[5]{-\frac{n}{2}\pm\sqrt{4m^5-n^2}}$.

III. **Solution by H. C. WILKES, Skull Run, West Virginia; and A. H. BELL, Hillsborough, Illinois.**

Factoring, etc., $(x^2+2m)(x^2+3m)=m^2-\frac{n}{x}$. Let $x=n$; then $x=n$, $\pm\sqrt{-m-1}$, $\pm\sqrt{1-2m}$, which will be the five roots.

Or $(x^2+3m)(x^2+2m)=(m+\sqrt{\frac{n}{x}})(m-\sqrt{\frac{n}{x}}$. Assuming $x^3+3m=\sqrt{\frac{n}{x}}$, $x^3+2m=m-\sqrt{\frac{n}{x}}$. $x=\sqrt{-\frac{3m}{2}}$, $m=2\sqrt{\frac{n}{x}}$; hence $x=\frac{4n}{m^2}$. Substituting $m=2\sqrt{\frac{n}{x}}$ for m in eq. 1, $x^5+10x^3\sqrt{\frac{n}{x}}+21n=0$. This can be developed, $x^{10}-58nx^5+441=0$. $\therefore x=\sqrt[5]{49n}$ or $\sqrt[5]{9n}$.

[The above is not strictly a solution, but affords a method of discovering integer roots, if any. The solu- of Professor Zerr is especially full and neat. EDITOR.]

Also solved by *F. P. MATZ.*

52. **Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.**

In how many ways can we arrange 12 friends of the MONTHLY, around a table, so that; (1) the editors may never be together, (2) Matz and Halsted may never be apart, and (2) Zerr and Ellwood may always have Gruber betwixt them?

Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

I. Considering one editor in position the other may occupy 9 places; but the first editor may take 12 places, and therefore the two take 108 positions. For each of these places the remaining nine mathematicians may be seated in $\lfloor 9$ ways, making $108\lfloor 9$ ways altogether.

II. If Matz and Halsted are never apart we may consider them as an element to be arranged as *each* of the other individuals. We then have $\lfloor 11$ ways of arranging them without regarding the *internal* arrangement of the group; this may be arranged in two ways. We, therefore, have $2\lfloor 11$ as the number of arrangements.

III. By the same reasoning as in the last case we have the number of arrangements $=2\lfloor 10$.

NOTE.—No solution of problem 53 has as yet been received. The published solution of problem 49, in last issue, should have been credited to Prof. J. H. Grove, Howard Payne College, Brownwood, Texas.

PROBLEMS.

59. **Proposed by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.**

Demonstrate the identity $2^{2n+1}\frac{d^n}{dx^n}\left(x^{n+\frac{1}{2}}\frac{d^{n+1}}{dx^{n+1}}e^{\sqrt{x}}\right)=e^{\sqrt{x}}$.

60. **Proposed by Professor C. E. WHITE, Trafalgar, Indiana.**

Prove that every algebraic equation can be transformed into another equation of the same degree, but which wants its n^{th} term.

61. **Proposed by J. A. CALDERHEAD, A. B., Superintendent of Schools, Limaville, Ohio.**

Given $x^2+x\sqrt{xy}=10$, and $y^2+y\sqrt{xy}=20$ to find x and y by quadratics.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

40. **Proposed by F. P. MATZ, D. Sc., Ph.D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.**

The closed portion of the curve known as "The Cocked Hat," equation

$$x^4+x^2y^2+4ax^2y-2a^2x^2+3a^2y^2-4a^3y+a^4=0,$$

revolves around the axis of y. Find the *campanulate* volume generated. If the same portion of the curve revolve around the axis of x, find the *fusiform* volume generated. Also, determine the area of this closed portion of the curve.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas; W. C. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts; and the PROPOSER.

Solving the equation for x^2 we get $x^2=\pm\frac{1}{2}y\sqrt{y^2+8ay}-\frac{1}{2}(y^2+4ay-2a^2)$.

$\therefore$ The campanulate volume generated by the area MPA_1QNM is

$$V=\frac{\pi}{2}\int_0^a(y\sqrt{y^2+8ay}-y^2-4ay+2a^2)dy+\frac{\pi}{2}\int_0^{\frac{1}{2}a}(y\sqrt{y^2+8ay}+y^2+4ay-2a^2)dy.$$

$$=\frac{\pi}{2}\Big[\tfrac{1}{3}(y^2+8ay)^{\frac{3}{2}}-2a(y-4a)\sqrt{y+8ay}+32a^3\log\Big\{y+4a+\sqrt{y^2+8ay}\Big\}-\tfrac{1}{3}y^3-2ay^2+2a^2y\Big]_0^a+\frac{\pi}{2}\Big[\tfrac{1}{3}y^3+2ay^2-2a^2y+\tfrac{1}{3}(y^2+8ay)^{\frac{3}{2}}-2a(y+4a)\sqrt{y^2+8ay}+32a^3\log\Big\{y+4a+\sqrt{y^2+8ay}\Big\}\Big]_0^{\frac{1}{2}a}$$

$$=\tfrac{4}{3}\pi a^3(12\log 3-13).$$

[ZERR, MATZ, and BLACK.]

From the equation we get $y=\dfrac{2a(a^2-x^2)\pm(a^2-x^2)\sqrt{a^2-x^2}}{x^2+3a^2}$.

$\therefore$ The fusiform volume is

$$V=8a\pi\int_0^a\frac{(a^2-x^2)^2\sqrt{a^2-x^2}}{(x^2+3a^2)^2}dx=8\pi a^3\int_0^{\frac{1}{2}\pi}\frac{\cos^6\theta d\theta}{(4-\cos^2\theta)^2},\text{ where } x=a\sin\theta,$$

$$=8\pi a^3\left[\frac{17\theta}{2}-\frac{44\sqrt{3}}{3\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\tan\theta\right)+\frac{\cos\theta\sin\theta}{2}+\frac{8\sin\theta\cos\theta}{9\cos^2\theta+12\sin^2\theta}\right]_0^{\frac{1}{2}\pi}$$

$$=4\pi^2a^3\left(\frac{17}{2}-\frac{44\sqrt{3}}{9}\right)=\frac{2}{9}\pi^2a^3(153-88\sqrt{3}).$$

Also area is $A=2\displaystyle\int_0^a\frac{(a^2-x^2)\sqrt{a^2-x^2}}{x^2+3a^2}dx=2a^2\int_0^{\frac{1}{2}\pi}\frac{\cos^4\theta d\theta}{4-\cos^2\theta}=2a^2\Bigg[\frac{8\sqrt{3}}{3}$

$\tan^{-1}\left(\dfrac{2}{\sqrt{3}}\tan\theta\right)-\dfrac{9\theta}{2}-\dfrac{\sin\theta\cos\theta}{2}\Bigg]_0^{\frac{1}{2}\pi}=\dfrac{1}{6}\pi a^2(16\sqrt{3}-27).$

[ZERR, and MATZ.]

Or, fusiform volume$=2\pi\displaystyle\int_0^a(y_1^2-y_2^2)dx=16a\pi\int_0^a\frac{(a^2+x^2)^{\frac{5}{2}}dx}{(x^2+3a^2)^2}$

$$=16a^3\pi\int_0^{\frac{1}{2}\pi}\frac{\cos^6\theta d\theta}{(4-\cos^2\theta)^2}$$

$$=16a^3\pi\int_0^{\frac{1}{2}\pi}\left(\frac{\cos^6\theta}{4}+\frac{2\cos^8\theta}{4^3}+\frac{3\cos^{10}\theta}{4^4}+\frac{4\cos^{12}\theta}{4^5}+\ldots\ldots\right)d\theta$$

$$=8\pi^2a^3\left\{\frac{1.3.5}{2.4.6.4^2}+\frac{2.1.3.5.7}{2.4.6.8.4^3}+\frac{3.1.3.5.7.9}{2.4.6.8.10.4^4}+\ldots\ldots\right\},\text{ since}$$

$$\int_0^{\frac{1}{2}\pi}\cos^{4m}xdx=\frac{1.3.5\ldots\ldots\ (2m-1)}{2.4.6\ldots\ldots\ldots\ldots(2m)}\cdot\frac{\pi}{2}.$$

Area of closed portion$=2\displaystyle\int_0^a(y_1-y_2)dx+4\int_0^a\frac{(a^2-x^2)^{\frac{3}{2}}dx}{x^2+3a^2}$.

Let $x=a\sin\theta$, $A=4a^2\displaystyle\int_0^{\frac{1}{2}\pi}\frac{\cos^4\theta d\theta}{4-\cos^2\theta}=4a^2\int_0^{\frac{1}{2}\pi}\Bigg(\frac{\cos^4\theta}{4}+\frac{\cos^6\theta}{4^2}+\frac{\cos^8\theta}{4^3}$

$+\dfrac{\cos^{10}\theta}{4^4}+\ldots\ldots\Bigg)d\theta=2a^2\pi\left\{\dfrac{1.3}{2.4.4}+\dfrac{1.3.5}{2.4.6.4^2}+\dfrac{1.3.5.7}{2.4.6.8.4^3}+\ldots\ldots\right\},$

which series is also convergent.

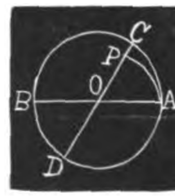
[BLACK.]

41. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

A railroad turn-table 100 feet long is balanced upon a pivot in the center of a circular track 100 feet in diameter. How far does a man walk who starts at one end of the table and walks, at a uniform rate, the entire length of the table in the same time that the table makes two revolutions, if the table starts to turn at the same time the man starts to walk?

Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

It is the purpose of this solution to find how far the man moves in space, if he always walks on the same line CD until across.



Let $OA=a$, $OP=r$, $\angle COA=\theta$, the velocity of C around the track n times the velocity of P along CD, P being the man's position at any time. Then $n.PC=\text{arc } AC$ $=a\theta$, $\therefore\ PC=\dfrac{a\theta}{n}$.

$\therefore\ r=a-PC=a-\dfrac{a\theta}{n}, =\dfrac{a(n-\theta)}{n}$. $\therefore\ r=a(n-\theta)/n$

is the equation of the man's path; also, $ds=\sqrt{(dr)^2+r^2(d\theta)^2}$, but $(dr)^2=$ $\dfrac{a^2}{n^2}(d\theta)^2$. $\therefore\ ds=\pm\dfrac{a}{n}\sqrt{1+(n-\theta)^2}\,d\theta$. $\therefore\ s=-\dfrac{2a}{n}\int\sqrt{1+(n-\theta)^2}\,d\theta$, for whole length $=\dfrac{a}{n}(n-\theta)\sqrt{1+(n-\theta)^2}+\dfrac{a}{n}\log\{n-\theta+\sqrt{1+(n-\theta)^2}\}+C$, but $2na$ $=4\pi a$, $\therefore\ n=2\pi$, $\therefore\ r=a(2\pi-\theta)/2\pi$.

$$s=\frac{a}{2\pi}(2n-\theta)\sqrt{1+(2\pi-\theta)^2}+\frac{a}{2\pi}\log\{2\pi-\theta+\sqrt{1+(2\pi-\theta)^2}\}.$$

The limits of θ are 0 and 2π, and $a=50$. $\therefore\ s=50\sqrt{1+4\pi_2}+\dfrac{25}{\pi}\log\left\{2\pi+\sqrt{1+4\pi^2}\right\}$.

$\therefore\ s=338.303$ feet.

Similarly solved by *G. B. M. ZERR*.

43. Proposed by J. C. NAGLE, M. A., C. E., Professor of Civil Engineering, A. and M. College, College Station, Texas.

Show that the volume included between the surface represented by the equation $z=e^{-(x^2+y^2)}$ and the xy plane equals the square of the area of the section made by the zx plane, the limits of x and y being plus and minus infinity.

I. Solution by Professor J. SCHEFFER, A. M., Hagerstown, Maryland.

Changing to polar co-ordinates, the volume is $=\int_0^{2\pi}d\theta\int_0^{\infty}e^{-r^2}rdr=$ $\frac{1}{2}\int_0^{2\pi}d\theta=\pi$. The area is $2\int_0^{\infty}e^{-x^2}dx$. Putting in the Gamma Function

$\int_0^\infty e^{-z}z^{n-2}dz=\Gamma(n)$, $z=x^2$, $n=\frac{1}{2}$, we find $2\int_0^\infty e^{-x^2}dx=\Gamma(\frac{1}{2})=\sqrt{\pi}$, which proves the assertion.

II. **Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.**

Let V=the required volume ; A=the required area.

$$\therefore V=\int\int\int dxdydz=\int\int zdxdy=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{-(x^2+y^2)}\,dxdy.$$

$$\therefore V=\left[\int_{-\infty}^{+\infty}e^{-x^2}dx\right]\left[\int_{-\infty}^{+\infty}e^{-y^2}dy\right].\quad A=\int\int dxdz=\int zdy=\int_{-\infty}^{+\infty}e^{-x^2}dx.$$

But $\int_{-\infty}^{+\infty}e^{-x^2}dx=\int_{-\infty}^{+\infty}e^{-y^2}dy$. $\therefore V=\left[\int_{-\infty}^{+\infty}e^{-x^2}dx\right]^2=A^2$.

III. **Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.**

$z=e^{-(x^2+y^2)}$. Applying formula for volume, $V=\int\int zdydx$, we have

$V=4\int_0^\infty\int_0^\infty e^{-(n^2+y^2)}dydx$......(1). Also let $y=0$. Then $z=e^{-x^2}$ is the equation of section made by zx plane. Area$=2\int_0^\infty e^{-x^2}dx$........(2). Let this be equal to a......(3). Now put (1) in form of $V=4\int_0^\infty\int_0^\infty e^{-x^2}e^{-y^2}dydx$. Integrating with reference to y in accordance with (3), we have $V=2\int_0^\infty ae^{-x^2}dx=2a\int_0^\infty e^{-x^2}dx=a^2$, also in accordance with (3).

Professor William Hoover did not solve this problem but referred to Todhunter's Integral Calculus, Art. 204, where a good solution is given.

PROBLEMS.

49. **Proposed by B. F. BURLESON, Oneida Castle, New York.**

Find (1) in the leaf of the strophoid whose axis is a the axis of an inscribed leaf of the lemniscata, the node of the former coinciding with the crunode of the latter. Find (2) in a leaf of the lemniscata whose axis is b the axis a of an inscribed leaf of the strophoid, the node of the former also coinciding with the crunode of the latter.

50. Proposed by GEORGE LILLEY, Ph. D., LL. D., Principal of Park School, 329 East Second Street, N. Portland, Oregon.

A draw bridge, a feet in length, moves uniformly about a center axis. At the instant it began to open, a man stepped on the end; and, walking at a uniform rate in the straight line passing through its center, reached the opposite end just as it made n complete revolutions. Find the absolute path described by the man, and the ratio of his rate of motion in this path and the velocity of the end of the bridge. Apply the result to the case when $a=320$ and $n=2$.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

30. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

P is the lowest point on the rough circumference of a circle in a verticle plane at which a particle can rest, friction being equal to the pressure; to find the inclination of the radius through P to the horizon.

Solution by the PROPOSER.

If μ=the coefficient of friction, R=the normal reaction of the curve, μR= the friction, $=R$ by the problem. $\therefore$ $\mu=1$.

W being the weight of the particle, we have, resolving along the tangent and radius through P,

$$W\sin\phi=\mu R\text{......}(1).$$

$$W\cos\phi=R\text{......}(2).$$

These give $\tan\phi=\mu=1$, or $\phi=\dfrac{\pi}{4}$.

Excellent solutions of this problem were received from *PROFESSORS ALFRED HUME, O. W. ANTHONY*, and *E. L. SHERWOOD*.

31. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

A perfectly elastic, but perfectly rough mass M and radius R, rotating in a verticle plane with an angular velocity of ω, is let fall from a height, a, upon a perfectly elastic, but perfectly rough horizontal plane. Determine the motion of the body after striking the plane. What will be its ultimate motion?

Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics in the University of Mississippi, University P. O., Mississippi.

Let the angular velocity, ω, be in the direction of the motion of the hands of a clock.

Let ω' and v' be, respectively, the angular velocity, and the horizontal velocity of the center of the sphere, after the first impact.

The impulsive action at the point of contact is, then, Mv'.

The change in the angular momentum being equal to the moment of the impulse,

$$\tfrac{2}{5}MR^2(\omega-\omega')=Mv'R.$$

The surfaces being perfectly rough, there is no slipping and

$$v'=R\omega':$$

$$\therefore\ \tfrac{2}{5}(\omega-\omega')=\omega',$$

$$\omega'=\tfrac{2}{7}\omega;$$

$$v'=\tfrac{2}{7}R\omega.$$

v'' and ω'' representing horizontal and angular velocities after second impact,

$$M(v''-v')=\text{impulsive friction},$$

$$\tfrac{2}{5}MR^2(\omega''-\omega')=-M(v''-v')R,$$

$$\tfrac{2}{5}(\omega''-\omega')=\omega'-\omega'',$$

$$\omega''=\omega'=\tfrac{2}{7}R\omega,\text{ and } v''=R\omega''=\tfrac{2}{7}R\omega.$$

The sphere moves on in an endless series of equal parabolas, with constant angular velocity and constant horizontal velocity, reaching the height a after every rebound.

Solutions of this problem were also received from Professors Zerr and Anthony. One or both of these solutions will appear in the next issue of the MONTHLY.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

30. Proposed by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

A and B are two integers, A consisting of $2m$ figures each being 1, and B consisting of m figures each being 4. Prove that $A+B+1$ is a square.

I. Solution by H. W. DRAUGHON, Olio, Mississippi, and O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

Each of the integers is a Geometrical Series.

$A=1+10+100+\text{etc.}$, to $2m$ terms, $=\frac{1}{9}(10^{2m}-1)$.

$B=4+40+400+\text{etc.}$, to m terms, $=\frac{4}{9}(10^m-1)$.

$A+B+1=\frac{1}{9}(10^{2m}-1)+\frac{4}{9}(10^m-1)+1=\frac{1}{9}(10^{2m}+4.10^m+4)$

$=\{\frac{1}{3}(10+2)\}^2$.

II. **Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas, and H. C. WILKES, Skull Run, West Virginia.**

$A=\frac{9}{16}B^2+\frac{1}{2}B$ as is shown by the following: Let $B=444$.

$\therefore\ \frac{9}{16}B^2+\frac{1}{2}B=111111$. This is true for any value of B.

$$\text{Hence } A+B+1=\tfrac{9}{16}B^2+\tfrac{1}{2}B+1=\left(\frac{3B+4}{4}\right)^2=B^2.$$

$\therefore\ A+B+1=(333......334)^2$, the number within the parenthesis consists of m figures. Let A_1 be an integer consisting of m figures all 1's.

Then $B^2=(333......334)^2=(B+1+A_1)^2$.

Also solved by *M. A. GRUBER* and *J. SCHEFFER.*

31. **Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.**

How many scalene triangles, of integral sides, can be formed with an altitude of 12? How many isosceles triangles?

I. **Solution by ARTEMUS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington D. C.**

1. To find right-angled triangles having one leg=12.

Let x=the required leg and $x+a$=the hypothenuse; then $(x+a)^2-x^2=2ax+a^2=12^2=144$; whence $x=\dfrac{144-a^2}{2a}$.

It is easily seen that a must be even, and that it cannot exceed 10; but as x must be integral a can only be 2, 4, 6, or 8.

Take $a=2$, then $x=35$; take $a=4$, then $x=16$; take $a=6$, then $x=9$; take $a=8$, then $x=5$. Hence there are four right-angled triangles having one leg =12, viz: 12, 35, 37; 12, 16, 20; 12, 9, 15; 12, 5, 13.

2. Any two right-angled triangles, p, c, a; p, b, d, can be combined in two different ways to form a scalene triangle, giving the triangles a, b, $c+d$; a, b, $c-d$. Hence the four right-angled triangles found above can be combined two and two in two different ways to form scalene triangles; therefore there are twelve such triangles which have an altitude of 12, as follows: 13, 14, 15; 20, 37, 51; 15, 20, 25; 15, 37, 44; 13, 37, 40; 13, 20, 21; 13, 15, 4; 20, 37, 19; 15, 20, 7; 15, 37, 26; 13, 37, 30; 13, 20, 11.

There can be only four isosceles triangles with integral sides having an altitude of 12, viz: 13, 13, 10; 15, 15, 18; 20, 20, 32; 37, 37, 70.

II. **Solution by A. H. BELL, Box 184, Hillsboro, Illinois.**

We evidently require to find two □ numbers whose difference shall be equal to any given number. Let x=the side of the lesser square, and d=to two unequal factors $=ab$, $a>b$; let $x+b=$the greater square.

Then $(x+b)^2-x^2=ab$, and $x=\frac{a-b}{2}$, $x+b=\frac{a+b}{2}$.

The unequal factors of the difference $(12)^2$ are 2×72, 4×36, 6×24, 8×18; these give for sides of squares in the formula, and complete the following right-angled triangles, in the order of altitude, base and hypotenuse: 12, 5, 13; 12, 9, 15; 12, 16, 20; 12, 35, 37.

By doubling the base of each will give four isosceles, and by adding and subtracting the bases from each pair will give 12 scalene triangles.

III. Solution by the PROPOSER.

All scalene △'s are rt. △'s or are the sum or the difference of two rt. △'s of equal altitudes. The △'s of this problem are restricted to △'s of integral sides having an altitude of 12.

We first find the rt. △'s of integral sides having an altitude of 12. These are four in number: 12, 5, 13; 12, 35, 37; 12, 9, 15; and 12, 16, 20.

Then, by *sum* and *difference*, we form combinations by twos by joining their equal altitudes. It will readily be seen, if $n=$the number of rt. △'s of a given altitude, that the number of combinations each by *sum* and by *difference* of twos is the sum of the series, $n-1$, $n-2$, $n-3$......1. The sum of this series is $\frac{n(n-1)}{2}$. As there are two such series, the number of combinations is $n(n-1)$.

Adding to this the n rt. △'s, we find the total number of scalene △'s to be n^2, which is the square of the number of rt. △'s having the given altitude. Hence the number of scalene △'s of integral sides having an altitude of 12 is $4^2=16$.

All isosceles △'s of integral sides are the union of two equal rt. △'s by joining the altitudes. There are as many isosceles △'s of integral sides having a given altitude as there are rt. △'s of integral sides having the given altitude. Hence there are four isosceles △'s of integral sides having an altitude of 12.

Also solved by *O. W. ANTHONY*, *H. W. DRAUGHON*, *G. B. M. ZERR*, and *WILLIAM HOOVER*.

32. Proposed by A. H. BELL, Box 184, Hillsboro, Illinois.

Decompose into its prime factors the number 549755813889.

Solution by the PROPOSER.

To find the factors of $2^{39}+1=549755813889$. The old masters have demonstrated that prime factors of a^n+1 must be of the general form of $2nx+1$. Suppose we take $a^{mn}+1$, mn odd, the factors of mn are m, n, 1; then the prime divisors will be of form $a^{mn}+1$, a^n+1, and $a+1$. Divide out these factors; the $\sqrt{\text{balance}}$ will show the limit of the trial divisors which must be of the general form $2mnx+1=$to prime form of factors$=8mnx+1$ and $8mnx+(6mn+1)$, if these will not or if they do divide the balance, we conclude the balance to be a prime number.

Solution of $2^{39}+1=549,755,813,889\div$by divisors (prime) $2^{13}+1$, 2^3+1, and $2+1$; then $3^2.2731.(22366891).\sqrt{22366891}=4620$ limit of divisors of the form prime $8mnx+1$ and $8mnx+(6mn+1)=312x+1$ and $312x+235$, and they are 313, 547, 859, 937, 1171, 1249, 1483, 1873(2731)3121, 3433, 4057, 4603 to limit, none of which will divide the balance, hence 22366891 is prime.

$\therefore$ factors are $3^2\times2731\times22366891$.

No solution of Problem 33 has been received.

PROBLEMS.

43. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find the series of integral numbers in which the sum of any two consecutive terms is the square of their difference.

44. Proposed by A. H. HOLMES, Box 963. Brunswick, Maine.

The hypotenuse of a right-angled triangle ABC, right-angled at A, is extended equally at both extremities so that $BE=CD$. Draw AD and AE. Find integral values for all the lines in the figure thus made.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

NOTE ON AVERAGE AND PROBABILITY WITH REFERENCE TO THE SOLUTIONS OF PROBLEM 26, pp. 282-83, AND 327-28.

By ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington D. C.

I WILL remark at the outset that, unfortunately, mathematicians are not agreed as to the method of solving certain problems in Average and Probability. The difference of opinion in some cases relates to the interpretation of the meaning of the problem, and in others to the quantity that should be considered as the independent variable, and between what limits taken, and again as to whether the "points" are uniformly distributed along a certain line or over a certain surface, etc.

If points be uniformly distributed on a line, the *number* of points is proportional to the length of the line; and if points be uniformly distributed over a surface, the number of points is *proportional* to the area of the surface, etc.; but if the points be *not* uniformly distributed, then the line or surface can not be taken as a *true* measure of the number of points.

Problem 26. "Find the average of all right-angled triangles having a given hypotenuse."

Prof. Matz' first solution, p. 82, would be correct if he had taken the limits of x from 0 to h instead of from 0 to $\frac{1}{2}h\sqrt{2}$. He supposes one of the legs to increase uniformly from 0 to $\frac{1}{2}h\sqrt{2}$, or till the legs become equal; but this assumption does not give *all* the triangles because, while x *increases* uniformly from 0 to $\frac{1}{2}h\sqrt{2}$, $\sqrt{(h^2-x^2)}$ does not decrease uniformly from h to $\frac{1}{2}h\sqrt{2}$. The limits should be 0 and h, for if one leg varies uniformly from 0 to h *all possible* right-angled triangles will be generated, and the number of the triangles will be proportional to h.

If he had taken x from 0 to h in his first solution, and θ from 0 to $\frac{1}{2}\pi$ instead of from 0 to $\frac{1}{4}\pi$ in the second, he would have obtained in both solutions the result $\frac{1}{6}a^2$, which I believe to be correct.

In the second method of solution, adopted by Prof. Zerr and others, and approved by the Editor, it is assumed that one of the *acute angles* varies uniformly, and that the *number* of triangles is proportional to the semicircumference whose diameter is the given hypotenuse.

The vertices of the right angles of all possible right-angled triangles having a given hypotenuse a will be posited on a semicircumference whose diameter is a, but *will not* be uniformly distributed thereon; hence the semicircumference *can not* be taken as the true measure of the number of triangles.

I most emphatically dissent from the conclusion announced by the Editor in the last line of p. 328. The last paragraph of the note is sound down to the last line, but it does not by any means necessarily follow from any statement made therein that "the solutions leading to the result $\frac{a^2}{2\pi}$ are the *correct* and *only* solutions of the problem." I hold that the solution given by Prof. Anthony on p. 283, and previously given by myself in the *Mathematical Magazine*, leading to the result $\frac{1}{6}a^2$ (misprinted $\frac{1}{3}a^2$ in the first line of the Editor's note), is the *true* solution of the problem.

The conception of a triangle is from its sides; and if we cause one of the legs to take all possible values from 0 to a it is very clear to me (and ought to be to every one) that all possible right-angled triangles having that hypotenuse will be formed.

The problem as proposed is *definite* as the Editor correctly states in his note, and requires the "average area of *all* the right-angled triangles having a given hypotenuse"; but the solution which he asserts the "*correct* and *only*" one restricts the triangles to those having the vertices of their right angles uniformly distributed on the semicircumference whose diameter is the given hypotenuse, and therefore is not a solution of the problem proposed, but of the following problem, viz: Required the average area of the right-angled triangles having a given hypotenuse and the vertices of their right angles uniformly distributed on the semicircumference whose diameter is the given hypotenuse.

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

Why was the letter π chosen to represent $\frac{c}{d}$?

LOTTIE SMITH, Houston, Miss.

I have several problems, which though solved by quadratics, have one positive root and no other. I have also several problems, which can only be solved by quadratics, which have two roots, one real and one imaginary, not withstanding that "imaginary expressions enter an equation by pairs," which at present I will not disclose. The following problem is from Bell's Algebra (Chamber's Edition Course): "Given $\sqrt{2x^2-2})=3x-5$. $x=3$, or $\frac{9}{7}$." The $\frac{9}{7}$ fails to verify the equation. Can another root besides 3 be found that will?

R. GREENWOOD, Morris, Ill.

COMMENTS ON PROBLEM 11—GEOMETRY.

What does the gentleman do with the parts of the circle outside of his own central circle and the seven circles he gives to his seven children? If he does not "retain" it, he must think that these pieces will suit his wife.

W. F. BRADBURY,
Cambridge Latin School, Cambridgeport, Mass.

When the condition of the problem is satisfied, one of the seven equal circular farms will be concentric with the original farm. This condition is, therefore, incompatible with the (insinuated) condition that the gentleman shall retain for himself an area about the center of the original farm. The "problem" is merely a puzzle.

L. E. PRATT, Tecumseh, Neb.

On page 249 of Wentworth's *College Algebra*, we find the author conclude that $\underline{|0}=1$, *i. e.*, factorial zero is equal to unity. On page 246 the definition of *factorial* is given: $\underline{|n}=n(n-1)(n-2)\ldots\ldots1$, *i. e.*, factorial n is equal to the product of all the natural numbers from n to 1 inclusive. If n were 3, we would have $\underline{|3}=3.2.1$; if $n=8$, then $\underline{|8}=8.7.6\ldots\ldots1$. So for any other number. If therefore 0 is to be one of them, it must submit to the same law.

$\therefore$ $\underline{|0}=0\ldots\ldots1=0.1=0$! This would show that factorial zero, if it has any meaning at all, must be equal to zero. But *factorial zero* is not comprised within the definition given by the author: by that definition the first factor of the product is the number given (n), the last is unity, (1), which therefore excludes 0. The mistake made by the author in arriving at the result consists in disregarding the factor 0 in one of the terms of a fraction. From the formula

$C_{n,r}=\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{\lfloor r}$ is derived, by multiplying each term by $\lfloor n-r$, the formula: $C_{n,r}=\frac{n(n-1)\ldots\ldots(n-r+1)(n-r)\ldots\ldots 1}{\lfloor r\times(n-r)\ldots\ldots 1}$ ..[A],

or $C_{n,r}=\frac{\lfloor n}{\lfloor r\ \lfloor n-r}$. Now if in this latter we make $n=r=1$ we obtain

$$C_{1,1}=\frac{\lfloor 1}{\lfloor 1\ \lfloor 1-1}=\frac{1}{1.\lfloor 0}=\frac{1}{\lfloor 0}\ldots\ldots(B).$$

[But as the number of combinations made of one element with one in the group is one, we also have

$$C_{1,1}=1.\quad \therefore 1=\frac{1}{\lfloor 0}\ .\ \therefore\ \lfloor 0=1.]$$

But suppose we make the substitution in formula [A];

$$C_{n,r}=\frac{n(n-1)\ldots\ldots(n-r+1)(n-r)\ldots\ldots 1}{\lfloor r\,(n-r)\ldots\ldots 1}$$

where $(n-r+1)=(1-1+1)=1$.

$$C_{1,1}=\frac{(1)(n-r)\ldots\ldots 1}{\lfloor 1\,(n-r)\ldots. 1}=\frac{1}{1}\cdot\frac{\lfloor n-r}{\lfloor n-r}=\frac{1}{1}=1 \text{ and not: }\frac{1}{\lfloor 0}.$$

The factor which becomes 0 in the denominator also occurs in the numerator and is to be cancelled. The error made as in [B], consists in neglecting this factor in the numerator but retaining it in the denominator.

Oscar Schmiedel,
Bethany College, Bethany, West Virginia.

ANSWERS TO QUERIES IN MONTHLY FOR MARCH, 1894, (VOL. 1, NO. 3, P. 103.)

By Prof. John N. Lyle, Fulton, Mo.

I. Whether Lobatschewsky's theorem 4 is "sound" or not depends upon what shall be regarded as "sound" in geometry. If the assumption that a plane is the surface of a sphere and that two straight lines drawn therein perpendicular to a third do intersect is sound; then Lobatschewsky's theorem 4, since it contradicts this assumption, must be unsound. Otherwise, two propositions that contradict each other may both of them be sound. Again, if the soundness of Euclid's propositions 27 and 28, Book 1. is granted, that of Lobatschewsky's theorem 4 must also be conceded, since it is a legitimate corollary of those propositions.

II. Lobatschewsky's theorem 4 which reads as follows: "Two straight lines perpendicular to a third never intersect, how far soever they be produced" contradicts flatly the assumption that these perpendiculars do intersect, no matter where the intersection is supposed to occur. The *fact* and not the *place* of supposed intersection constitutes the contradiction. Von Staudt's assumption that two straight lines perpendicular to a third have "at infinity a common point" contradicts proposition 27, Book 1. of Euclid's Elements, and hence can not be in harmony with it. Euclidean space cannot be extended to any point of intersection of the two perpendiculars under notice for the good and sufficient reason that those perpendiculars do not and can not intersect in that space.

III. No. IV. Yes.

V. No, for the reason that it involves contradiction. By definition every straight line having two ends is finite. Hence, to affirm that such a line is infinite in length is to attribute to it contradictory attributes. No infinite straight line can be drawn between two points located in space and geometrical science does not concern itself with what is supposed to occur or not to occur outside of space. Juggling with algebraical symbols can not alter the cold, hard facts of the Euclidean geometry.

VI. In his theorem 16 Lobatschewsky is *studiously silent* as to whether he regards the boundary line itself as a *cutting* or a *not-cutting* line. In his theorem 33, however, he uses this language—"hence not only does the distance between two parallels decrease (Theorem 24), but with the prolongation of the parallels towards the side of the parallelism this at last wholly vanishes. Parallel lines have therefore the character of asymptotes." From this it appears that Lobatschewsky holds that the distance between asymptotes and their curves "at last wholly vanishes."

VII. In theorems 32 and 33 Lobatschewsky exhibits without disguise his use and interpretation of the symbols 0 and ∞, and his speculative opinions respecting geometrical data that dominate his thinking and thus determine his conclusions. The reason assigned by Lobatschewsky for his conclusion that the distance between parallels decreases and "at last wholly vanishes" is that $s'=0$ for $x=\infty$ in the formula $s'=se^{-x}$. There is nothing novel, brilliant or profound in manipulating algebraical symbols in such fashion. It is in fact a familiar game of analytical sophistry more than two hundred years old played in the school of Leibnitz with 0 and ∞ as dice. In his theorem 32 Lobatschewsky informs us that "A circle with continually increasing radius merges in the boundary line." He further says that "one may also call the boundary line *a circle with infinitely great radius.*"

When Lobatschewsky rejected Euclid's axiom 12 and accepted in its stead a straight line as the circumference he evidently strained at a gnat and swallowed a camel. In Lotze's Metaphysics, Part II., Vol. I., pages 290 and 291 the following extract is found: "A finite arc of a circle, of course, becomes perpetually more like a straight line as the radius of the circle to which it belongs is increased; but the whole circle never comes to be like one. However infinitely great we may conceive the radius as being, nothing can prevent us from conceiving it to complete its rotation around the center, and till such rotation is completed we have no right to apply the conception of a circle to the figure which is generated: discourse about a straight line which being in secret a circle of infinite diameter, returned into itself, is not a portion of esoteric science, but a proof of logical barbarism. Just the same is shown by phrases about parallel lines which are supposed to cut each other at an infinite distance. They do not cut each other at any finite distance, and as every distance when conceived as attained would become finite again, there is simply no distance at which they do so; it is utterly inadmissible to pervert this negation into the positive assertion that in infinite distance there is a point at which intersection occurs."

EDITORIALS.

We were compelled to omit the Department of Geometry in this issue because of lack of sorts, and the Miscellaneous Department because this number has now grown far beyond its proper limits.

No pains will be spared on the part of the editors to make Vol. III. of great value to all its readers. To this end, we trust that we may have the coöperation of all of our old contributors and that of many new ones.

Professor E. L. Sherwood should have been given credit for solving Problem 46, Department of Geometry. Editor Colaw and Prof. Cooper D. Schmitt each sent a solution of Problem 54, Department of Arithmetic, but too late for credit in the proper place.

A correspondent who has a large collection of mathematical autographs and MSS. will exchange duplicates with any other who is interested in the same line. Professor Finkel will put this correspondent in communication with any one who will send his address.

In order that we may increase the subscription list of the MONTHLY, we invite each of our old subscribers to take advantage of the following offer:

To any old subscriber sending us the names of three new subscribers, and six dollars, we will send THE AMERICAN MATHEMATICAL MONTHLY one year as a premium. This offer ought to quadruple the number of our subscribers.

While much is being said in the literary world about endowing magazines, what is wrong with making the MONTHLY an example of endowed periodicals?

This year a great friend of the MONTHLY and a Professor of Mathematics in an eastern college, invested $60. in extra copies. If one hundred of our subscribers would donate $50. towards an endowment fund, they would be entitled to a perpetual subscription and the MONTHLY saved from the fate of its predecessors,—discontinuance in the course of a decade or two.

This number completes the second volume of the MONTHLY, and though its success in the two years during which it has been issued has not been what we hoped for, it has not been altogether discouraging. We are encouraged by words from various mathematicians of great eminence that the MONTHLY is growing in influence and favor. We believe this to be true. A glance at our list of contributors will show that it includes the best mathematicians in America. Having the support of the ablest mathematicians of this country, the MONTHLY should continue to appear each month during the year that is now upon us. The editors have, therefore, no thought of discontinuing its publication, and we trust that we may have the earnest support of all of our old subscribers in the still further enhancing of its worth. In the January number we shall use a better quality of paper and thus improve its appearance. We have on hand a number of very excellent articles from leading mathematicians which will appear during the coming year. Dr. Halsted will continue his translation of Saccheri's geometry, and Dr. Miller will continue his articles on Substitution Groups. Dr. Moore has furnished an article on An Interesting System of Quadratic Equations, which will appear in the January number. Prof. Zerr has furnished an article on the Centroid of Plane Areas, the first part of which will also appear in the January number. A great many other papers of interest and importance from prominent mathematicians may be expected. The January number will contain an interesting biography of the great Russian Mathematician, Wolfgang Bolyai, by Dr. Halsted. Other biographies of noted mathematicians will be published during the year.

A great many of our subscribers are in arrears on subscription for Vol. I. and Vol. II. We shall be greatly obliged if those owing us will kindly remit at once, as we are much in need of funds. Please send money by Draft or Post-office Money Order to B. F. Finkel, 1320 Washington Avenue, Springfield, Mo.

BOOKS AND PERIODICALS.

High School Mathematical Teaching and Text-Books. A monograph from the *Inland Educator.* By Robert J. Aley, A. M., Professor of Mathematics in the University of Indiana, Bloomington, Indiana.

In this little pamphlet of 20 pages, Professor Aley has given some good hints on the teaching of Mathematics in the High School. He blames the teacher, the text-book, or both for the hatred so many boys and girls have for Mathematics. He says, page 13, "This feeling is not natural to the normally constituted mind. Psychologists and educators generally tell us that the mind ought to find pleasure in mathematical exercises. There is no subject in which the student can so early begin making discoveries for himself. The mere beginner in geometry can make and solve exercises that would have made Appollonius famous. This hatred for mathematics must then in general be the fault of the teacher, or of the text-book, or of both. Whichever it may be, it is in the province of the teacher to remove."

Wilkes' Rules of Multiplieation. By H. C. Wilkes, Skull Run, West Virginia.

In this little pamphlet of 16 pages, Mr. Wilkes has given a number of rules for the rapid multiplication of two numbers. He says, page 1, "To be expert in multiplying, three things are essential: 1st, To be able to multiply any number by a single digit, operating from left to right; 2nd, To know instantly from memory the product of any two numbers each less than 20; 3rd, To be able to add mentally and quickly any two numbers each less than 100." We give a single Rule: *How to mentally see the product of any two uumbers in the "teens."* Example: 19×16

$$\begin{array}{r} 19 \\ 6 \\ 54 \\ \hline 304 \end{array}$$

Place the unit figure of one of the numbers under the other number, and then place the product of the unit figures as shown, and add all together.

Laboratory Methods of Teaching Mathematics in Secondary Schools. By Adelia R. Hornbrook, A. M., Teacher of Mathematics in High School, Evansville, Indiana, and author of Concrete Geometry for Common and Grammar Schools. Pamphlet, 16 pages. Chicago: American Book Co.

In this little pamphlet, Mrs. Hornbrook has given some timely recommendations on the Laboratory Method of Teaching Mathematics from a psychological standpoint. She proves conclusively that the Laboratory Method which has worked wonders in the Department of Natural Science, can also be made to produce equally good results in Mathematics if sufficient care is taken on the part of the teacher. The term "Laboratory Method" as applied to teaching Mathematics means the method of independent personal investigation on the part of the learner under the leadership of a teacher who furnishes only the necessary aids to interpretation. The pamphlet is well worth a careful reading.

Plane and Solid Geometry. By Wooster Woodruff Beman, Professor of Mathematics in the University of Michigan, and David Eugene Smith, Professor of Mathematics in the Michigan State Normal School. 8vo. cloth and leather back, 320 pp. Boston: Ginn & Co.

In the last issue of the MONTHLY, we announced that Drs. Beman and Smith had written a Geometry and that something new along the line of Geometry might be expected. The book is now ready and we are quite sure that it will meet with public favor. It has a number of very strong points in its favor. We can only mention a few of these. (1) It invests the geometry of the Ancients with something of the spirit of Modern Mathematics; (2) Many terms that are not new but are rarely found in similar works have been freely used and thus the student is made familiar with a nomenclature that is very essential to the study of Modern Higher Mathematics; (3) Methods of attack are suggested early, and at the end of Book III. are treated with considerable fullness; (4) Historical notes, which have a tendency to relieve the monotony of class routine and awaken interest in even the most stupid and indifferent student, are frequently inserted; (5) A biographical table containing the names of forty-two mathematicians, Ancient and Modern, who have been instrumental in shaping the course of Mathematics, is appended; and (6), a table of Etymologies is also appended.

The value of a table of Etymologies can not be over estimated. The student coming in contact with new terms will naturally be interested in the etymology, a study of which will fix the meaning in his mind. I believe that it would have proved even more valuable had the etymology been given in connection with the term when it is used in the text.

The principles of Duality and Continuity are illustrated and explained. The book contains 783 exercises for original work. The book is a most admirable one and we take pleasure in recommending it to any who are seeking a good text on geometry.

B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year. Single number, 10 cents. Irvington-on-the-Hudson, New York.

No one ever thought of introducing so expensive a feature as lithographic color work in the days when the leading magazines sold for $4.00 a year and 35 cents a copy. But times change, and the magazines change with them. It has remained for *The Cosmopolitan*, sold at one dollar a year, to put in an extensive lithographic plant capable of printing 320,000 pages per day (one color). The January issue presents as a frontispiece a water-color drawing by Eric Pape, illustrating the last story by Robert Louis Stevenson, which has probably not been excelled even in the pages of the finest dollar French periodicals. The cover of *The Cosmopolitan* is also changed, a drawing of page length by the famous Paris artist Rossi, in lithographic colors on white paper takes the place of the manilla back with its red stripe. Hereafter the cover is to be a fresh surprise each month.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year. Single number, 25 cents. The Review of Reviews Co., New York.

During the closing weeks of 1895 the daily papers have published an extraordinary amount of interesting and important news. It is worth something to the busy newspaper reader to have this mass of information taken up, arranged, digested and reviewed in a calm and intelligent manner. The *Review of Reviews* performs this service very efficiently every month. The number for January, 1896, is especially strong in this respect. The editorial department called "The Progress of the World," is distinguished for its able handling of national and international topics of the hour. In fact, the *Review* occupies a unique position as a truly "international magazine." Its soundly "American" stand on the Venezuelan question is significant.

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